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August, 1960

Papers

DISCUSSION

(over)

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Proceedings of the American Society of Civil Engineers

REINFORCED CONCRETE SHEAR WALL ASSEMBLIES

By Jack R. Benjamin,[1] M. ASCE, and Harry A. Williams,[2] F. ASCE

SYNOPSIS

This paper gives the results of an investigation of one-story and two-story reinforced concrete shear wall assemblies. Tests were made on four one-story models with parallel shear walls connected by a reinforced concrete diaphragm. Loads were applied to the diaphragm, and three separate diaphragm tests were then made. Following these tests, two one-story specimens with face walls were tested. Finally, two two-story models were tested.

Theoretical studies were made for all tests and practical design approximations are suggested.

INTRODUCTION

This paper is concerned with a part of the investigation of strength and behavior of shear walls conducted between 1951 and 1956 at Stanford University, Calif. Other parts of the investigations have been previously published.[3,4,5]

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3 "The Behavior of One-Story Reinforced Concrete Shear Walls," by Jack R. Benjamin and Harry A. Williams, Journal of the Structural Division, Trans. ASCE, Vol. 124, 1959, pp. 669-708.

4 "The Behavior of One-Story Brick Shear Walls," by Jack R. Benjamin and Harry A. Williams, Journal of the Structural Division, Proc. ASCE, Paper No. 1723, ST 4, 1958.

5 "Behavior of One-Story Reinforced Concrete Shear Walls Containing Openings," by Jack R. Benjamin and Harry A. Williams, Journal of the American Concrete Institute, No. 5, Vol. 30, No. 1958, pp. 605-618.

The behavior of individual shear walls can be predicted within engineering accuracy.[3,4,5] Structures containing assemblies of shear walls connected by floor and roof diaphragms are most complex in their behavior. The investigations reported herein were designed to study the basic problems in the prediction of over-all structural response to lateral loads caused by blast. The conclusions apply equally well to wind and earthquake loads.

Three types of simple model structures were studied. These models were of approximately one-quarter scale. They did not correspond to any actual structures but, rather, were used to investigate various theoretical approaches to the problem. The mathematical studies considered these models as structures in themselves. While some scale effect may occur in translating the results to full-size structures, it is believed that such influences will be within the normal variations encountered with shear wall structures.

Four one-story parallel walled structures without face walls were tested. These were designated as specimens SD-2 to SD-5, inclusive. Model SD-2 contained two walls and a connecting diaphragm, and models SD-3, 4, and 5 had three parallel walls, also with connecting diaphragms. Three separate diaphragm tests were made in addition to determine the diaphragm properties.

Following these tests, two one-story specimens with face walls were tested. Model SD-6 had two identical shear walls, and specimen SD-7 had three different shear walls. Loads were applied to the diaphragm in such a manner as to produce large torsional forces. The influence of foundation distortion was studied extensively.

Finally, two two-story models were tested. These specimens, SD-8 and SD-9, were identical except for loading. Model SD-8 was loaded symmetrically, and model SD-9 eccentrically. These models had three walls in each story and no face walls.

EXPERIMENTAL PROCEDURES

All models were partially precast flat on the floor. The precast elements were then assembled in the testing jig and the remaining elements cast in place. In general, all walls were precast and all diaphragms were cast in place. All assemblies were tested in a specially designed shear jig. A model under test is shown in Fig. 1. Three individual diaphragms were tested in a 200,000-lb testing machine.

Type I portland cement was used in the concrete mix with a small amount of additive to improve workability. Precast elements were generally cured for one or two weeks before final assembly. Tests were conducted when the cast-in-place concrete was at a strength of approximately 3,000 psi. The precast elements were then at a concrete strength of approximately 3,800 psi at the time of test. All reinforcing consisted of structural or intermediate grade steel.

The assemblies were loaded at the diaphragm level using calibrated hydraulic jacks. Dynometers were provided at the hold-downs. The particular construction details and test procedures will be described subsequently.

ONE-STORY PARALLEL WALL STRUCTURES

Four one-story models with parallel shear walls were tested. Model SD-2 had two shear walls, while the others, SD-3, 4, and 5, each had three walls. Three individual shear walls, SD-1a, b, and c, were tested first as control specimens.

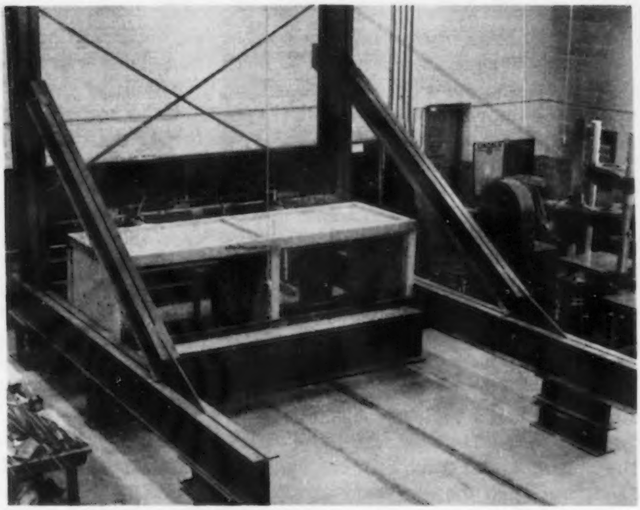


FIG. 1.—SHEAR JIG WITH SPECIMEN SD-5 UNDER TEST.

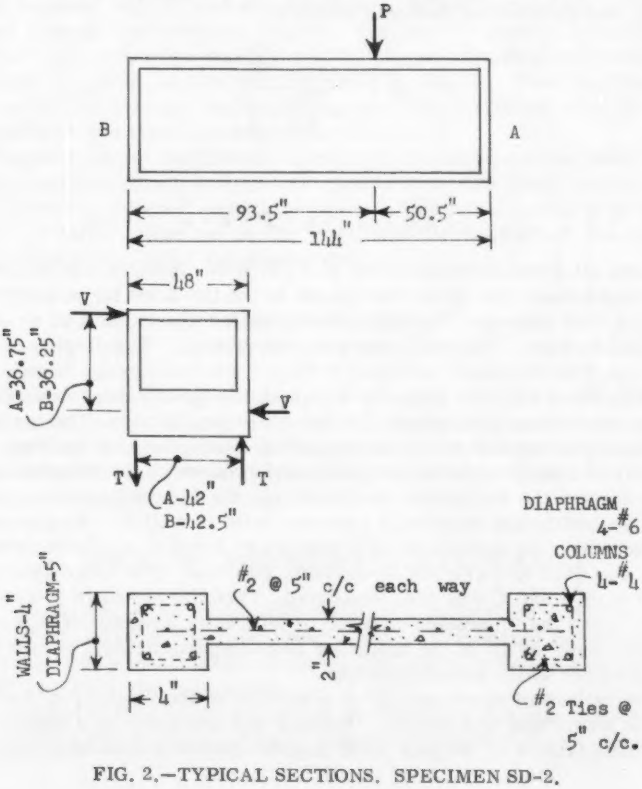


FIG. 2.—TYPICAL SECTIONS. SPECIMEN SD-2.

The investigation of Model SD-2 had two purposes. The first was to see whether a torsional couple resulting from shear wall warping had negligible influence on the diaphragm, as one would expect. The second was to determine whether roof diaphragms and shear walls behave in a similar manner.

Test results and approximate theoretical computations both showed that the torsion could be neglected. Diaphragm crack patterns, deflection measurements, and general behavior indicated that, theoretically, shear walls and diaphragms can be treated in a similar manner.

Three one-story assemblies, each having three parallel walls, A, B, and C, were then tested. Wall and diaphragm elements were the same as for SD-2, Fig. 2, and the finished model appeared as in Fig. 1. Equal forces, P/2, were applied to the diaphragms midway between the walls as shown by the hydraulic jack locations in the photograph.

The theoretical rigidities of the walls and diaphragms were computed on the assumption that the steel could be neglected and the concrete was uncracked. Each wall then had a rigidity of $R = 0.980$ E kips per in. of deflection. Each diaphragm segment had a rigidity of $D = 0.174$ E kips per in.

If the foundation is infinitely rigid, the effective wall rigidities are equal. However, the testing apparatus deforms, changing the effective wall rigidities. Symmetry indicates that the two end walls, A and C, will have identical rigidities. However, wall B will have a smaller rigidity than the end walls due to testing apparatus-distortion. If the shear in a wall is V, and P is the total applied load, the solution for end wall shear is

$$\frac{V_A}{P} = \frac{0.188 + \dfrac{D}{R_B}}{1 + \dfrac{2D}{R_B} + \dfrac{D}{R_A}} \qquad \text{. . . . . (1)}$$

Also,

$$V_A = V_C, \quad V_B = P - 2\,V_A \qquad \text{. . . . . (2)}$$

Specimen SD-3 failed prematurely at a tie-down. Results are not shown but the data emphasized that great variations in the tie-down force occurred due to the initial load take-up. The center wall carried almost no load up to an applied load of 20 kips. The wall cracking was typical. The diaphragm did not crack.

Assembly SD-4 differed from SD-3 in that the tie-down column rods were changed to No. 4 bars to eliminate further tie-down failure. The test of SD-3 showed that center wall B was slow to pick up load, probably because jig distortion allowed both horizontal movement and rotation of the foundation of this wall. To control the horizontal movement for SD-4, an adjustable wedge was placed at the horizontal foundation reaction point of wall B. As the test proceeded, the foundation movement of B relative to A and C was measured. Then the wedge was used to force the three walls into line. The foundation rotation of B relative to A and C was also measured. Then the tie-down rod nuts were adjusted to make the foundation rotation of wall B the average of that of A and C. The net result of all of the adjusting was to approximately insure that the foundations of the walls moved together.

Ultimate failure of specimen SD-4 occurred at the junction of the tension beam of the diaphragm and wall C. Failure was complete as a result of concrete and bond failure at the pour joint despite extensive doweling through this area.

The load-deflection and tie-down force curves are given in Fig. 3. The test procedure is evident in the tie-down force curves where the forces varied at each load as the center wall was adjusted. Individual wall and diaphragm load-deflection curves are given in Figs. 4 and 5. Shear walls cracked in the usual diagonal pattern.

The one striking feature of the experimental data is that the tie-down force curves are, essentially, continuous straight lines in both the cracked and uncracked regions. The slope does not change as individual elements crack and no load redistribution due to cracking is apparent.

When the experimental results are compared with the theoretical prediction based on a rigid foundation, the correlation is not good. If walls A and C are assumed to have a theoretical rigidity of 0.98 E, the effective rigidity of wall B can be computed. Using an average value of $V_A/P = 0.31$ from the experimental curves, R_B is found to be 0.37 E. Computations show the jig to be less than 10% as rigid as the shear walls. The effective value of R_B of 0.37 E is reasonable considering the large influence of small errors in adjustment.

After trying to adjust the movement of wall B of SD-4, it was decided to test assembly SD-5 without such adjustment but rather to carefully measure all such movements. SD-5 was identical to SD-4 except that wall A had three times as much panel steel. Thus the elastic rigidities would remain identical but their cracked rigidities would differ.

The test of SD-5 proceeded very well up to an applied load of 84 kips when the joint between wall C and the diaphragm failed completely. Wall load-deflection curves are shown in Fig. 6. The curves show a greater cracked rigidity for wall A than for walls B and C. The relationship between individual wall shears and total applied load is given in Fig. 7. Note the constancy of slope from elastic through the cracked range. The variations in wall rigidities were insufficient to affect the shear distribution.

The relative shear distribution computed from theoretical wall rigidities and measured foundation movements agreed with that found experimentally, within the limits of experimental accuracy. Very small errors in measurement have a large influence on the computations because of the very large rigidity of the shear walls and diaphragms.

DIAPHRAGM TESTS

The tests of assemblies SD-2 to SD-5 inclusive produced very little reliable information as to diaphragm rigidity and strength. The rigidity information was of doubtful value because of the scatter of experimental data coupled with the fact that the desired component of diaphragm shear was not measured directly. Hence, three separate diaphragms, identical to those used for SD-2 to SD-5, were tested in various ways.

ONE STORY ASSEMBLIES WITH FACE WALLS

Two assemblies were tested with solid face walls front and rear. The assembly, SD-6, had two identical shear walls. The design is shown in Fig. 8. Load was applied eccentrically to the diaphragm in order to place significant torsional shears in the face walls.

The addition of face walls, for which the concrete was placed integrally with the transverse shear walls, greatly complicates and changes the theoretical

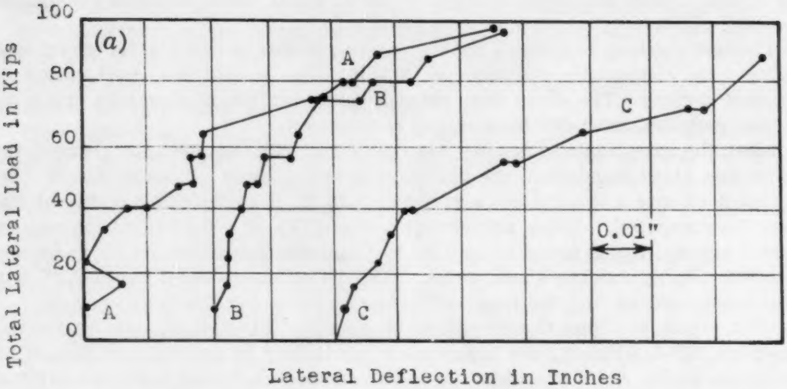


FIG. 3(a).—LOAD-DEFLECTION CURVES FOR SHEAR WALLS. SPECIMEN SD-4.

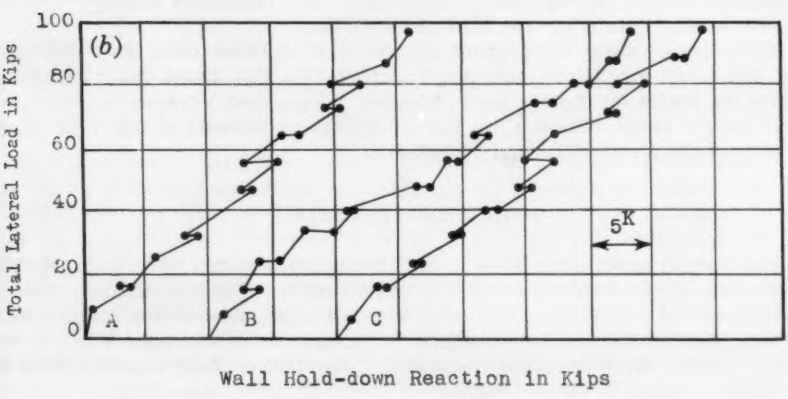


FIG. 3(b).—APPLIED LOAD VS. WALL HOLD-DOWN REACTIONS. SPECIMEN SD-4.

analysis. Consider the simple box type two-wall structure, having solid face walls, as shown in Fig. 9. Assume that the foundation is elastically supported but that the soil modulus is less at wall A than at wall B. Replace P by a force

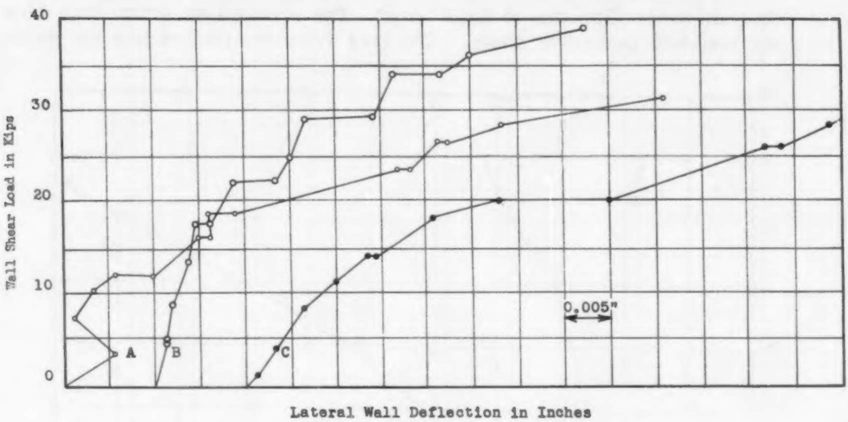


FIG. 4.—WALL SHEAR LOAD VS. WALL DEFLECTION. SPECIMEN SD-4.

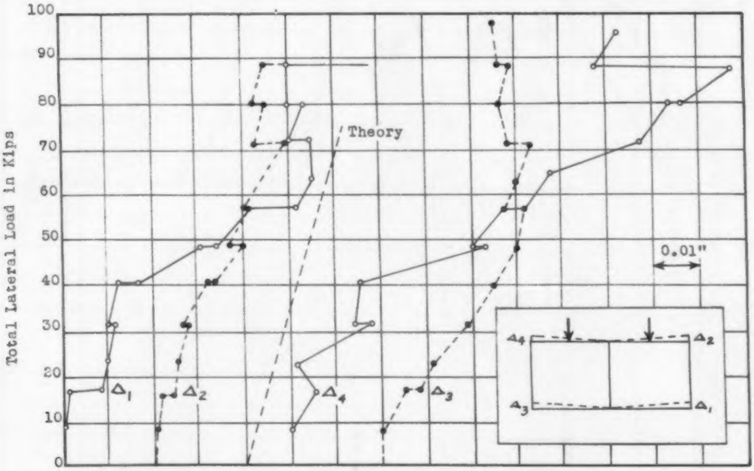


FIG. 5.—LOAD-DEFLECTION CURVES FOR DIAPHRAGM. SPECIMEN SD-4.

P on the center line and a torsional couple. Now separate the problem into three parts:

1. Direct translation of the diaphragm.
2. Foundation distortion.
3. Torsion of the assembly.

First consider only direct translation of the diaphragm without rotation and without foundation movement. The foundation is assumed to be infinitely stiff.

The structure is symmetrical and both shear walls are identical. Each transverse shear wall then carries half of the total load, P. The two horizontal reactions, F_A and F_B, are equal to $\frac{P}{2}$. The face walls have no shear stress and no distortion in the direction of their length. The over-all structure then acts as a vertical-box cantilever beam. The face walls are flanges and the walls

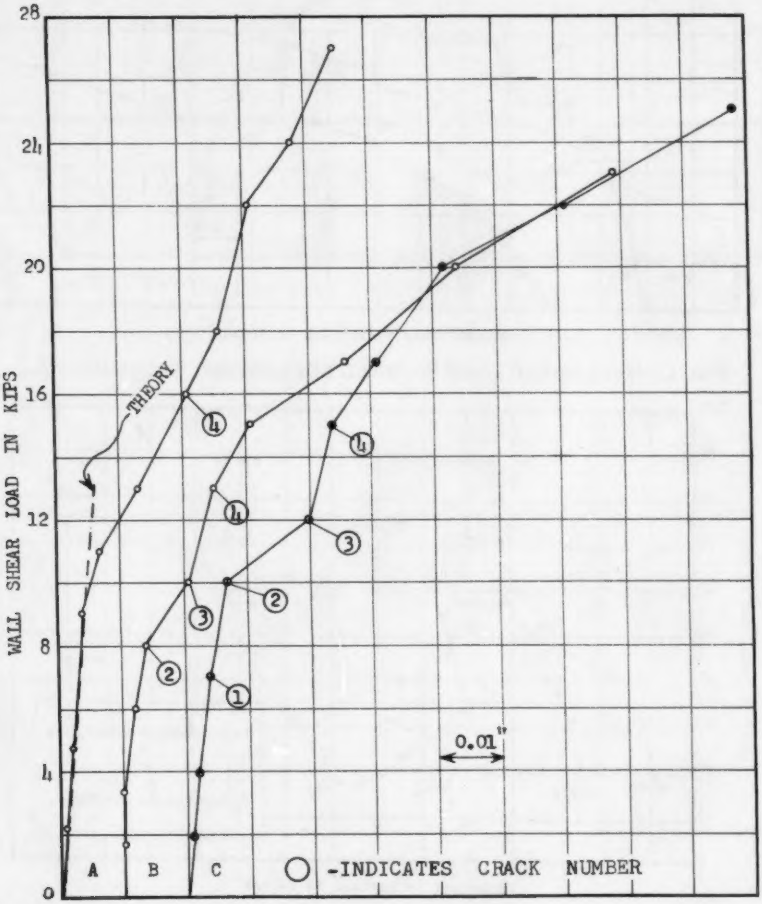


FIG. 6.—LOAD-DEFLECTION CURVES FOR SHEAR WALL. SD-5.

A and B, two identical webs. If simple theory is used, the face walls are essentially stressed only by bending while the end walls carry the shear.

Note that this solution is quite different from that obtained if the walls are not connected. If walls C and D are not placed integrally with A and B, they do not act as flanges of a beam but as separate cantilever beams in themselves. Thus the bending rigidity of the assembly is greatly reduced if the face walls

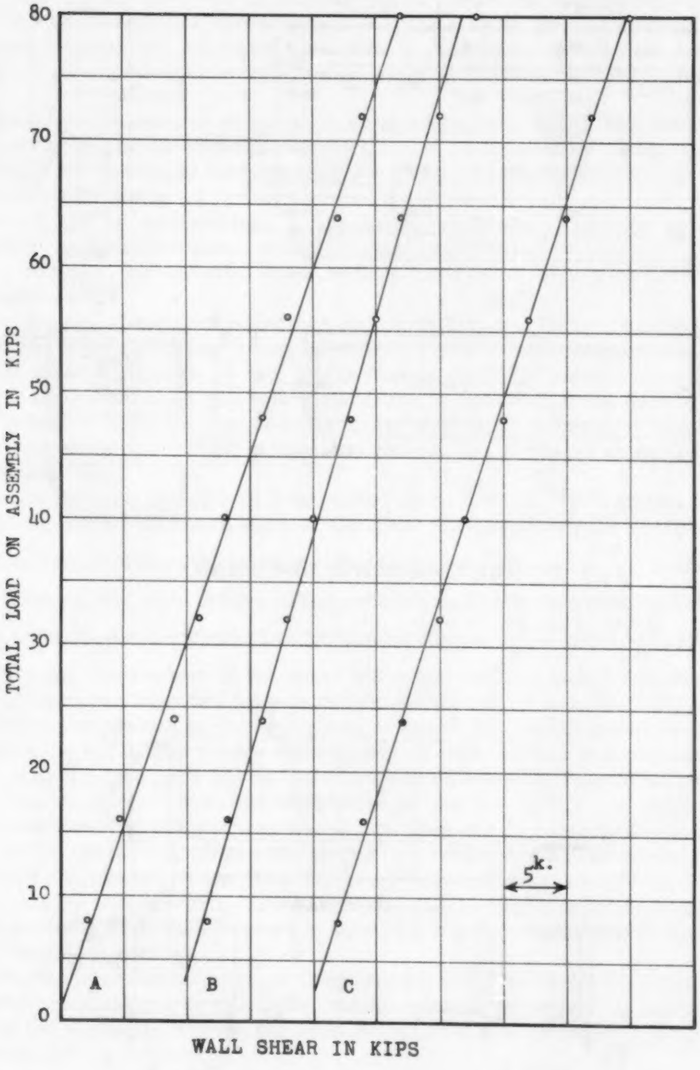


FIG. 7.—APPLIED LOAD VS. WALL SHEAR. SD-5.

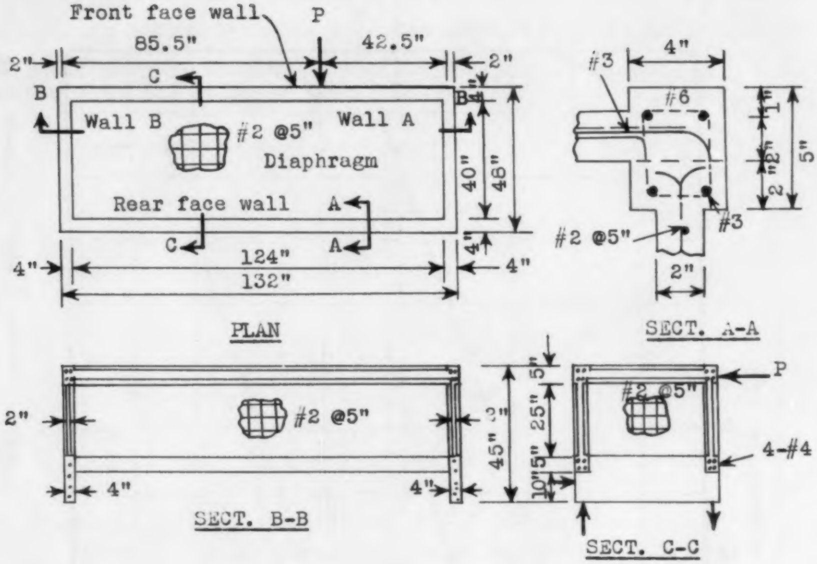


FIG. 8.—SPECIMEN SD-6 DETAILS.

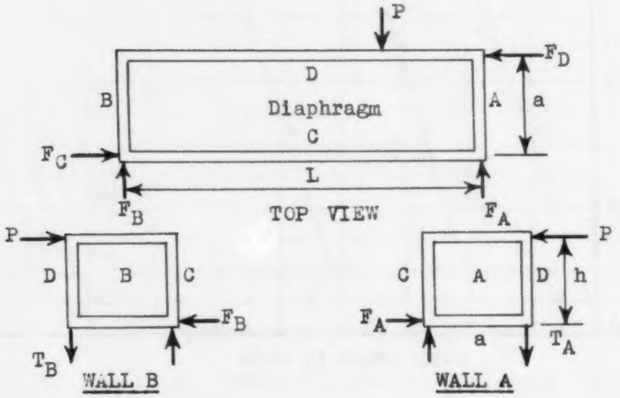


FIG. 9.—FORCES ACTING ON TWO-WALL BOX SPECIMEN SD-6.

are not connected to the end walls while the shearing rigidity is not greatly influenced by this assumption. With separate walls, the end walls act as vertical cantilever beams of almost rectangular section.

The solution for stresses is independent of the actual diaphragm rigidity. If the foundation deforms identically at A and B, the stress distributions are unchanged although the assembly distortion is modified. Note that the integral placing of concrete requires the consideration of the assembly as an integral unit.

Foundation distortion is of primary importance in all shear wall problems. With separate walls, the foundation distortion can be considered with each individual wall and combined directly with the individual wall rigidity to produce an effective rigidity.

With connected assemblies, a foundation distortion produces stresses throughout the structure unless either the diaphragm or foundation is very flexible. In contrast, an isolated shear wall is unstressed by a pure foundation movement.

Consider the simple box structure shown in Fig. 10. If the foundations of walls A and B translate and rotate identically, only a free body movement is involved. If the foundation of wall A translates a different amount from that of B, a free body rotation of the entire structure is involved. If the foundation of wall A rotates a different amount from that of wall B, all elements of the structure are stressed and eight reaction components are produced at the corners of the foundation.

Assume that one end of wall A is pushed up by a force, P_F. Vertical forces of equal magnitude are necessary at the other three corners for equilibrium. For simplicity assume P_F produces uniform shear stress, $\frac{P_F}{2\,t\,h}$, in the wall panels framing into each corner. Thus each end and face wall panel is subjected to a uniform shear stress. The horizontal forces, $\frac{P_F\,a}{2\,h}$ and $\frac{P_F\,L}{2\,h}$, are now necessary for equilibrium of the shear walls and the face walls, respectively. The diaphragm is placed in a state of pure shear stress, as were the wall panels.

If the diaphragm has no shear rigidity, it cannot be placed in a state of pure shear and the horizontal forces cannot exist. If these forces must be zero, P_F is zero and the structure has no resistance to differential foundation rotation other than the direct torsional resistance of the face walls and diaphragm. These torsional rigidities are so small that they can be neglected. Furthermore, if the horizontal forces act against a flexible foundation material, the foundation will change shape, (Fig. 11), these horizontal forces will not be produced, and P_F will be zero. The originally rectangular foundation plan must remain a rectangle if the structure is to have a significant resistance to differential foundation rotation.

If the diaphragm is infinitely stiff against shearing deformation, differential foundation rotation produces a pure rotation of the diaphragm. If the vertical edges of the assembly remain vertical, the original corner where P_F was first applied moves up a distance

$$\delta = \frac{P_F\,a}{2\,h\,t\,G} \qquad \text{(3a)}$$

or

$$\delta = \frac{P_F\,L}{2\,h\,t\,G} \qquad \text{(3b)}$$

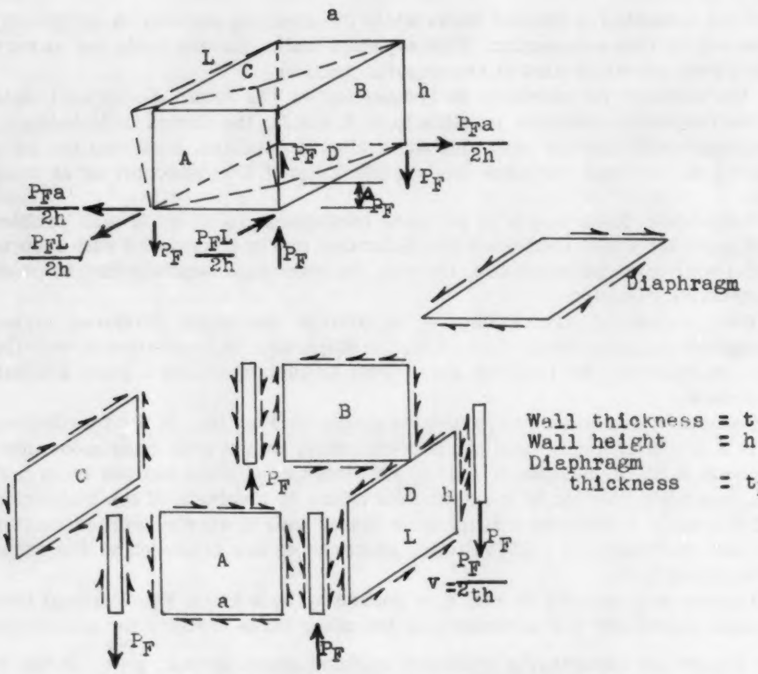


FIG. 10.—EQUILIBRIUM CONDITIONS RESULTING FROM APPLICATION OF FORCE P_F AT NEAR CORNER. SPECIMEN SD-6.

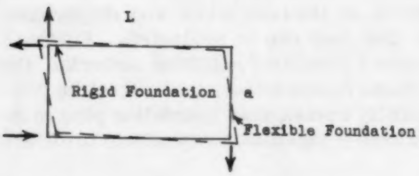


FIG. 11.—PLAN VIEW SHOWING DISTORTION OF RIGID AND FLEXIBLE TYPES OF FOUNDATIONS. SPECIMEN SD-6.

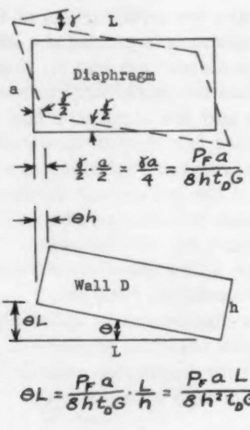


FIG. 12.—DISTORTION OF DIAPHRAGM AND MOVEMENT OF WALL D. SPECIMEN SD-6.

for end wall and face wall distortion, respectively. When the diaphragm rotates, the final deformation at the corner is the average of these two values or

$$\delta = \frac{P_F\,(a + L)}{4\,h\,t\,G} \quad \dots\dots\dots\dots\dots\dots\dots \quad (4)$$

If the diaphragm has a finite shear rigidity, its distortion does not change the magnitude of the stresses, but δ is increased. The unit shearing distortion in the diaphragm is

$$\gamma = \frac{P_F}{2\,h\,t_D\,G} \quad \dots\dots\dots\dots\dots\dots\dots \quad (5)$$

where t_D is the thickness of this member. Fig. 12 shows that this angular distortion produces a movement at the corner of the diaphragm of $\frac{P_F\,a}{8\,h\,t_D\,G}$. This movement involves a pure rotation of the face wall and a vertical deflection at its corner of $\frac{P_F\,a\,L}{8\,h^2\,t_D\,G}$. The total deflection of the assembly corner due to P_F is

$$\delta = \frac{P_F}{4\,h\,G}\left[\frac{a + L}{t} + \frac{a\,L}{2\,h\,t_D}\right] \quad \dots\dots\dots\dots\dots \quad (6)$$

If t_D is zero, δ becomes infinite, or P_F is zero. The differential foundation rotation of wall A is

$$\theta = \frac{\delta}{a} \quad \dots\dots\dots\dots\dots\dots\dots \quad (7)$$

Then

$$P_F = \frac{4\,h\,a\,G\,\theta}{\frac{a + L}{t} + \frac{2\,L}{2\,h\,t_D}} \quad \dots\dots\dots\dots\dots\dots \quad (8)$$

Once P_F is known, all other force quantities are readily found. If θ is a function of the total load on the assembly, Eq. 8 can be used to determine the direct relationship between P_F and the total assembly load. Evidently, the angle θ will depend on the characteristics of the supporting soil and the type of foundation.

Torsion must be considered if the line of action of the resultant of the applied loads on the structure does not coincide with the resultant horizontal reaction for pure translation. The stresses and deformations associated with the torsion are very complex. If conventional uniform torsion theory is used, the shear flow is constant in the wall panels around the diaphragm. With constant thickness, this means a constant uniform shear stress in all panels of

$$v = \frac{T}{2\,a\,t\,L} \quad \dots\dots\dots\dots\dots\dots\dots \quad (9)$$

Such a condition of pure uniform torsion involves warping of the cross-section unless the diaphragm is square in shape. Although warping is restricted, the

deformations involved are so small they do not in themselves preclude the application of uniform torsion theory.

Specimen SD-6 was tested extensively in order to prove or disprove the theory previously presented. Because considerable difficulty was experienced in the instrumentation of models SD-2 to SD-5, the entire instrumentation was revised for this specimen. Gages were mounted on an independent steel frame so as to measure the translation and rotation of the end wall foundations, the translation of the top of the end walls, and the load point on the diaphragm. In addition, the rectalinearity of the foundation was checked. Jacking was provided so that this or any desired condition could be obtained and also face wall shear could be measured.

Tie-down forces were measured using SR-4 gages on specially made tie-down rods. The horizontal end-wall reactions at the kick beam were also measured directly by jacks and pressure gages. This was necessary because the tie-down forces are derived from both face and end wall instead of end wall alone as with previous SD specimens having parallel shear walls only.

Three proof tests at low load were made first to check the operation of the equipment. After all items checked within reasonable accuracy, 14 test runs were made at 5 different load levels. Loads were increased from zero to maximum load and then reduced back to zero for each of the conditions shown in Table 1.

TABLE 1.—CONDITIONS OF LOADING

Series	Maximum Load Kips	Foundation Condition for Test Runs			
		I	II	III	IV
A	16	1	2	3	4
B	24	5	6	7	
C	40	8	9	10	
D	64	11	12	13	
E	96	14			

The foundation conditions listed in Table 1 represent four possible variations in foundation movement. In Condition I, the foundation was kept rectangular at all times. The foundation diagonals were measured at each load level and sufficient force F_D, (Fig. 9), was applied by jacking to make both diagonals change length the same amount in the same direction. Condition II was similar to Condition I. The changes in length of the foundation diagonals were measured under these conditions. Condition III is for zero F_D, or no shear in the face walls. Finally, Condition IV is the other possible statical limit when F_A equals zero. The latter condition was only studied at one load because of the danger of severely damaging the structure. Load Series E contains only one test because the ultimate load was reached in this test and the test was extended to include post-ultimate behavior with attendant large deformations. Thus Test 14 differs from the other 13.

Typical test data are shown in Figs. 13, 14, and 15. The horizontal wall reactions F_A and F_B are shown in Figs. 13 and 14 respectively. The dashed lines were computed. Note that both increasing and decreasing loads are given. The walls were cracked in this range. The observed values of F_D are given in Fig. 15 for tests 11 and 12. F_D was zero in test 13.

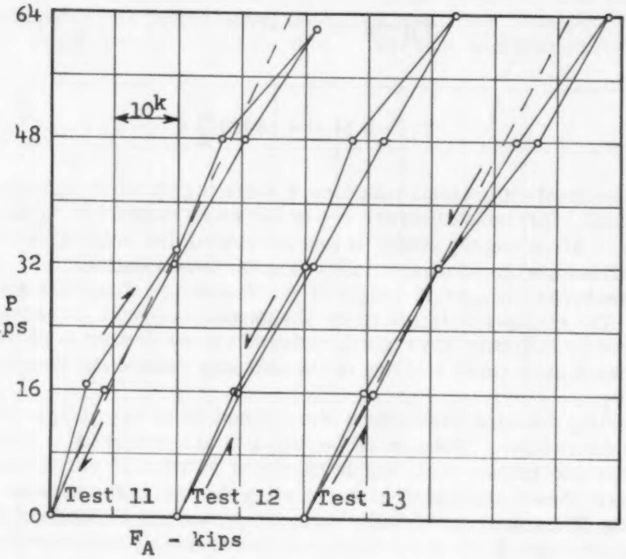


FIG. 13.—APPLIED LOADS VS. SHEAR IN WALL. A SPECIMEN SD-6.

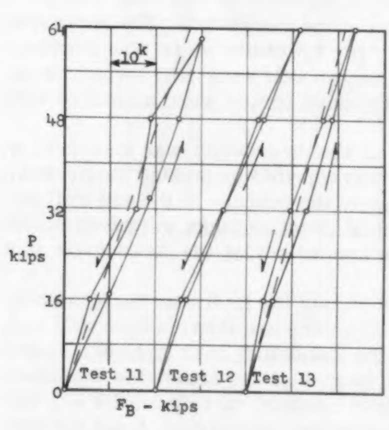


FIG. 14.—APPLIED LOAD VS. SHEAR IN WALL B. SPECIMEN SD-6.

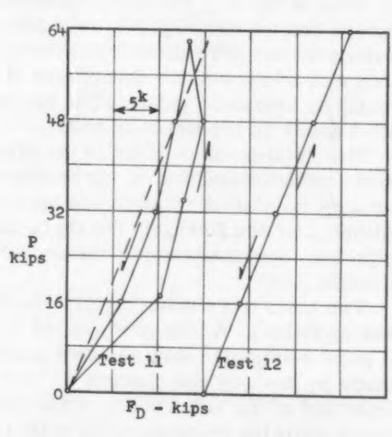


FIG. 15.—APPLIED LOAD VS. SHEAR IN WALL D. SPECIMEN SD-6.

With uniform torsion theory,

$$F_A = 0.58\ P - 18000\,\frac{a}{h}\,\theta \quad \text{(10a)}$$

$$F_B = 0.42\ P + 18000\,\frac{a}{h}\,\theta \quad \text{(10b)}$$

and

$$F_D = 0.24\ P + 18000\,\frac{L}{h}\,\theta \quad \text{(10c)}$$

The measured differential rotations, θ, were highly variable and nonlinear. The probable experimental errors are of the same magnitude as the measured quantities. After lengthy study, it was concluded that uniform torsion theory with no differential foundation rotation was the best solution.

The measured changes in length of the foundation diagonals are given in Fig. 16. The changes in length of the diagonals necessary to reduce the face wall shears by 50% from the rigid foundation values is seen to be very small. Thus, in practice, a small yielding of the soil may reduce the face wall shears appreciably.

The testing machine distortions were found to be very large compared to specimen distortions. Thus no deflection data are presented. Ultimate load was 72 kips and failure occurred in the upper portion of shear wall A. The relative wall shears carried by each resisting element again were not a function of cracking. Thus, wall A carried a constant proportion of the load for a given test condition regardless of the load or of the assembly disintegration.

The second face wall assembly, SD-7, differed considerably from SD-6. Details are shown in Fig. 17. Note that the three shear walls differ greatly from each other. Wall A contains a large opening, B has a solid, 2-in. panel, and C has a solid, 4-in. panel. The face walls and diaphragm are solid and similar to those of SD-6. The instrumentation was a duplicate of that used with SD-6 except that additional gages were necessary at the center wall. The horizontal wall reactions were measured directly by three hydraulic jacks. The resultant face wall shear and tie-down force at the center wall were also measured directly by hydraulic jacks. The end wall tie-down forces were measured with SR-4 gages on the tie-down rods.

The rotation of the foundation of each of the three walls was measured at each load level and that of the center wall was adjusted by jacking the tie-down until its foundation rotation was approximately the average of the end wall rotations. On the first test the three horizontal reaction jacks were continually adjusted, thus maintaining the ends of the foundations of the three walls in a straight line.

Two tests were made on this specimen. During the first test the procedure was as follows. A load was applied to the structure and then the face wall jack at point 1 (Fig. 17) was adjusted to bring the foundation back to a rectangular shape by making the changes in over-all lengths of the foundation diagonals equal and of the same sense. Then the center tie-down jack at point 3 was adjusted while the rectangularity of the foundation was maintained. A set of readings were then taken. After all items were recorded, the face wall shear was reduced to zero by releasing the jack at point 1 and the changes in length of foundation diagonals determined. All other items were also determined at this time. The face wall jack load was then re-applied, returning the specimen to

its original condition. All items were again recorded. The specimen was then ready for an increase in load. This was continued up to a load of 70 kips at which time the entire load was removed.

The procedure for the second test was the same as for the first except that the face wall shear was not varied at each load point. The foundation remained rectangular at all times during this test. This test was maintained into the

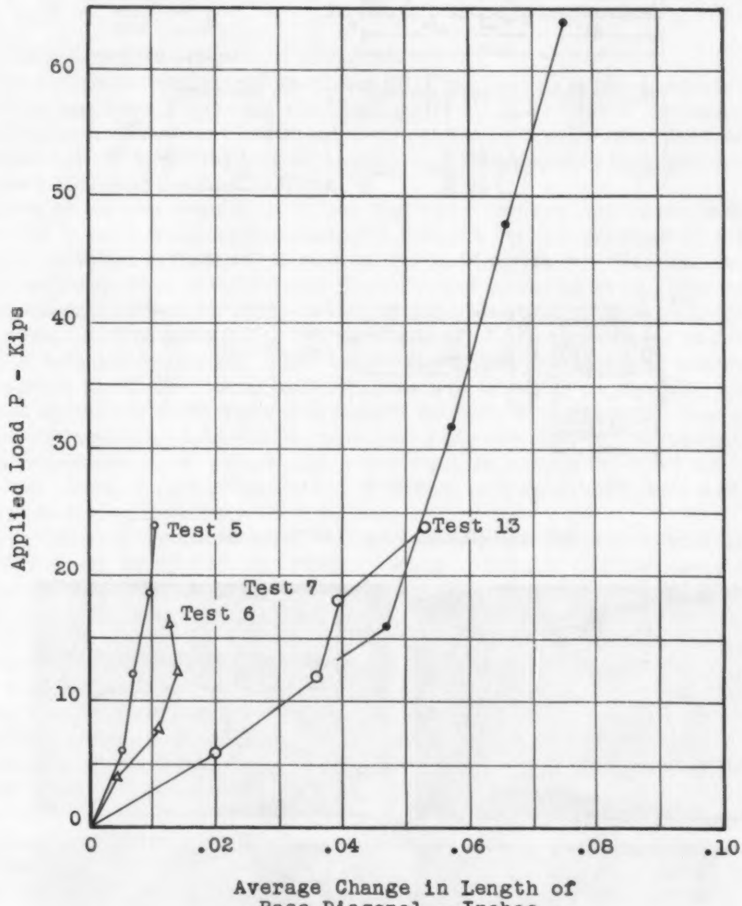


FIG. 16.—LOAD VS. BASE DIAGONAL CHANGES. SPECIMEN SD-6.

post-ultimate range. Some differential foundation rotation occurred during this test.

Photographs of specimen SD-7 are given in Fig. 18. For all these specimens, 93 kips was the maximum load. Fig. 18(a) is the rear face wall, Fig. 18(b) is the roof, Fig. 18(c) is wall A, Fig. 18(d) is wall B. Figs. 18(c) and (d) are for cycle 2. Wall C did not crack significantly. The loads at which the

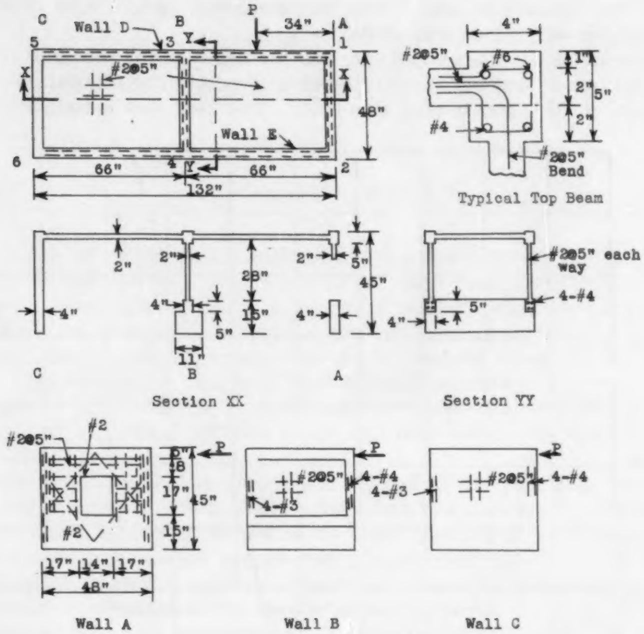


FIG. 17.—DETAILS OF SPECIMEN SD-7.

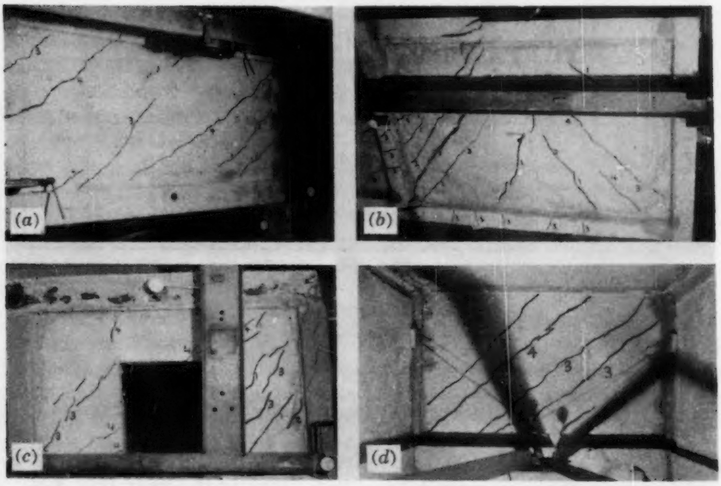


FIG. 18.—SPECIMEN SD-7.

various cracks occurred are as follows:

Crack Number	Load-Kips	Test Run
1	40	1
2	50	1
3	60	1
4	70	2

Typical test data are plotted in Fig. 19.

The theoretical analysis of specimen SD-7 can only be made on an approximate basis. If the diaphragm rigidity is zero, or if the base of the structure can deform freely from a rectangular shape, the face walls have zero shear load and walls A and B divide the load equally. This is one of the limiting conditions realized in the first test run.

Study of the test results discloses that when the face wall shear is made zero, F_C is zero, F_A is approximately 0.47P, and F_B is approximately 0.53P. This is excellent agreement between theory and experiment. Note that the results are independent of the relative rigidities and strengths of the shear walls.

The addition of a third shear wall to the structure makes the general theory somewhat more complex. The three walls in SD-7 are all different in design. Wall A contains an opening. The analysis of a wall containing an opening is complex in itself. Combining such elements with others in an assembly makes a most difficult problem and a true general analysis is impractical. Insofar as the overall assembly behavior is concerned, walls with openings can be replaced by an equivalent solid wall of such a thickness as to have the same effective rigidity. Such an approximation is reasonable as long as the actual wall can carry the forces involved without failure.

The equivalent solid wall for wall A is determined by first computing the rigidity of the actual wall including tributary face walls and then solving for the necessary thickness of a solid wall to produce the same overall rigidity. The equivalent solid web for wall A is 1.16 inches in thickness.

All three shear walls now have solid panels although each is of a different thickness. If the roof diaphragm is given a direct translation without rotation and the foundation does not distort, the shear taken by each wall is proportional to its relative thickness. The face walls are acting as flanges of a beam section having three webs of varying thickness. The unit shear stresses in each of the webs are equal and thus the shear carried by each wall is proportional to its relative thickness. For SD-7, the results indicated in Table 2 were obtained.

Pure translation of the diaphragm requires a properly located direct translational force. This force must pass through the shear center of the box section. The shear center is not located at the center of gravity of wall shear because direct translation of the diaphragm produces torsion in the assembly as well as bending and shear. The assembly is broken into elements in Fig. 20, using a unit applied load.

Consider wall A. If the total moment of inertia of the assembly is called I and the I of wall A including columns is I_A, then

$$M_A = M \frac{I_A}{\Sigma I} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

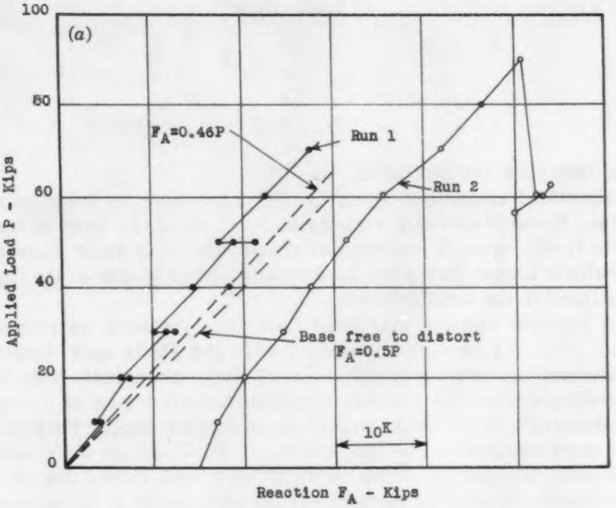


FIG. 19(a).—APPLIED LOAD VS. FOUNDATION REACTION F_A. SPECIMEN SD-7.

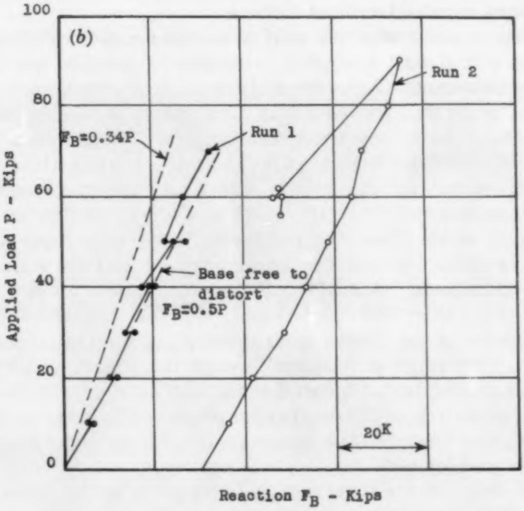


FIG. 19(b).—APPLIED LOAD VS. FOUNDATION REACTION F_B. SPECIMEN SD-7.

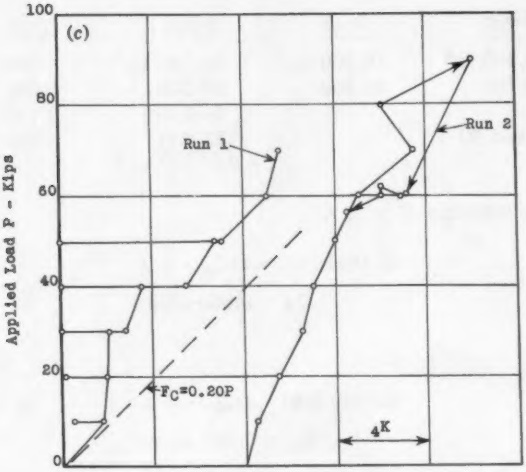


FIG. 19(c).—REACTION F_C VS. APPLIED LOAD. SPECIMEN SD-7.

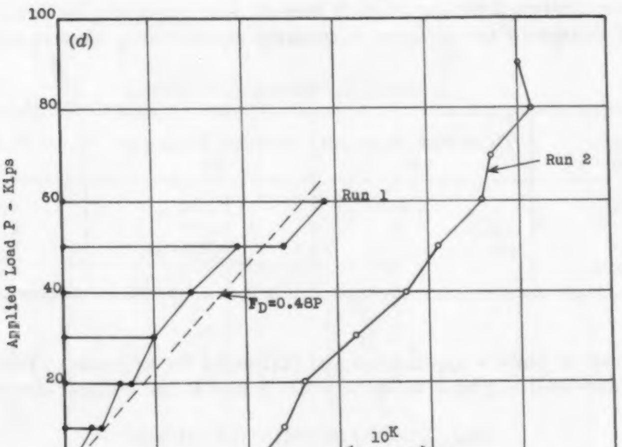


FIG. 19(d).—APPLIED LOAD VS. FOUNDATION REACTION F_D. SPECIMEN SD-7.

in which M = 28P, the total moment in the assembly at the foundation level. The following computations result if P = 1 kip:

Wall	I_{web}	I_{col}	Total	$I/\Sigma I$	$28\ I/\Sigma I$
A	6,200 in^4	15,500 $in.^4$	21,700 $in.^4$	0.068	1.9 kip in.
B	10,700	15,500	26,200	0.082	2.3
C			36,900	0.116	3.2
Face (Total D and E)			233,000	0.733	20.5
			ΣI = 317,800 $in.^4$		

Considering equilibrium of wall A;

$$(0.162)\ (28) = 44Q_A + 1.9$$

$$Q_A = 0.060 \text{ kips}$$

Wall C;

$$(0.559)\ (28) = 44Q_C + 3.2$$

$$Q_C = 0.283 \text{ kips}$$

The vertical force acting on each face-wall segment is equal to the moment divided by twice the center to center distance between them or

$$\frac{20.5}{(2)\ (44)} = 0.233 \text{ kips}$$

Now the vertical equilibrium of each face-wall segment can be completed. The face-wall segments are not now in moment equilibrium, thus requiring hori-

TABLE 2.—RESULTS FOR SD-7

Wall (1)	Thickness, in inches (2)	Relative Thickness (3)	Wall Shear (4)
A	1.16	0.162	0.162P
B	2	0.279	0.279P
C	4	0.559	0.559P
Total	7.16	1.000	1.000P

zontal forces or shears applied top and bottom of the segment. For example, the front face-wall segment between walls A and B has a shear-force of Q_{AB}.

$$28Q_{AB} = (30)\ (0.173) - (30)\ (0.060)$$

$$Q_{AB} = 0.121 \text{ kips}$$

The other shears can be computed, yielding the values shown in Fig. 20. The forces exerted on the top of the wall assembly at the diaphragm level are shown at the bottom of Fig. 20.

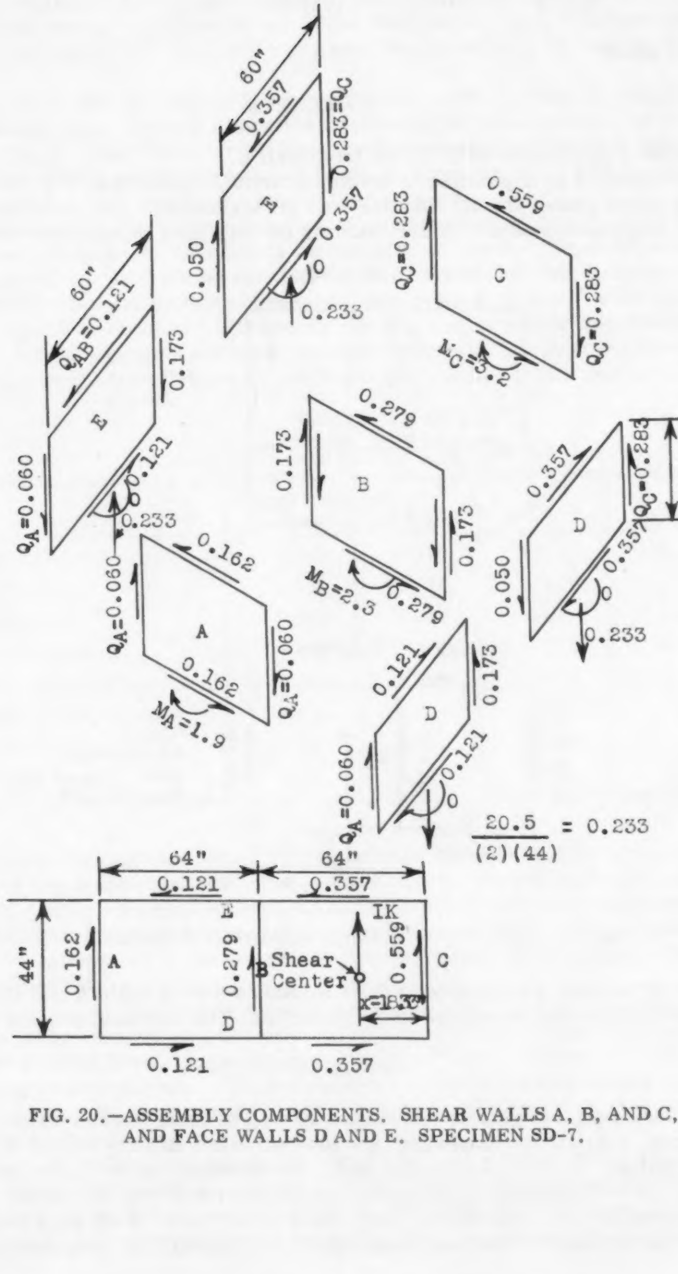


FIG. 20.—ASSEMBLY COMPONENTS. SHEAR WALLS A, B, AND C, AND FACE WALLS D AND E. SPECIMEN SD-7.

The applied load must be so located as to produce this force pattern. Taking moments about wall C,

$$(1)(x) = (0.279)(64) + (0.162)(128) - (0.121 + 0.357)(44)$$

From which

$$x = 18.33 \text{ in.}$$

Thus the shear center is 18.33 in. from wall C.

Obviously it is impossible to develop practical equations giving the locations of the shear center for all possible wall arrangements. Box structures, however, must be considered in this manner or very poor results are obtained.

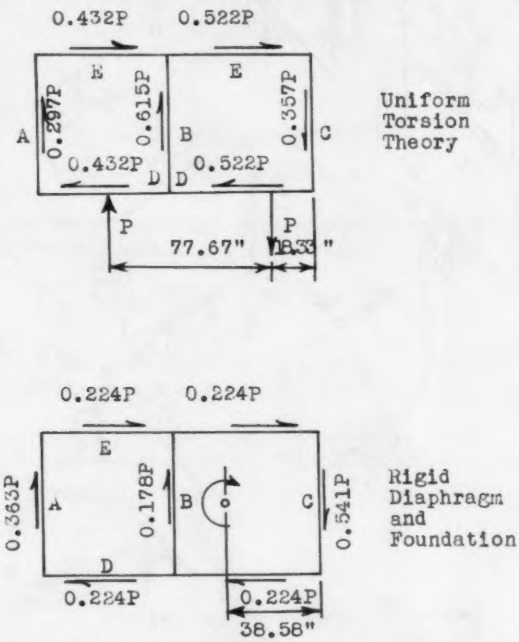


FIG. 21.—RESULTS OF TORSION ANALYSIS. SPECIMEN SD-7.

The actual load was applied to SD-7 midway between walls A and B. Thus, considerable torsion is induced in the structure. The torsional moment is then

$$P\left(\frac{64}{2} + 64 - 18.33\right) = 77.67P$$

The torsional analysis of this structure can be made on two different assumptions. First, if the diaphragm and foundation are flexible only to the point where uniform torsion theory can apply, the solution can be made using the uniform torsion approach from the theory of elasticity, Fig. 21. These forces are associated with a condition of pure shear in all walls. If all such deformation is concentrated in the roof diaphragm, the diagonals in each box element

change length. The diagonals change length in the box between A and B approximately 3×10^{-9} in. per kip of load, and between B and C approximately 1×10^{-7} in. per kip of load. Such magnitudes are very small compared to those produced by even a nominal shear stress in the diaphragm. Thus, it is reasonable to expect uniform torsion theory to apply even though warping is not completely free.

If both diaphragm and foundations are infinitely rigid, warping is completely restricted, and only shearing distortion is considered in the walls. The forces shown in Fig. 21 would then result. Such a solution involves rotation about the center of gravity of wall thickness, or 38.58 in. from wall C. Comparison of values discloses that the two solutions differ greatly.

The first test on SD-7 was conducted so as to have the rotation of the foundation of all three walls essentially identical at all times. Thus differential foundation rotation need not be considered. The second test run, however, does involve differential foundation rotation between walls A, C, and B such that the shear on wall B is reduced. The second run was performed in this manner in order to limit the time of testing to one eight-hour day. The first run gave excellent data and it was believed unnecessary to duplicate this portion of the test.

TABLE 3.—COMPARISON OF RESULTS

	F_A	F_B	F_C	$F_D = F_E$
Direct Translation	0.162	0.279	0.559	-0.478
Uniform Torsion	0.297	0.062	-0.359	0.954
Rigid Diaphragm	0.363	0.178	-0.541	0.448
Approx. Unif. Torsion	0.279	0	-0.279	0.810
Translation + Rigid Diaph.	0.525	0.457	0.018	-0.030
Translation + Unif. Torsion	0.459	0.341	0.200	0.476
Translation + Approx. U.T.	0.441	0.279	0.280	0.332
Experimental Slopes	0.40	0.42	0.18	0.50

Theoretical values are shown by the dashed lines in Fig. 19, expressing the results of these computations. The uniform torsion theory method of solution was used. Table 3 compares the results using uniform torsion, rigid diaphragm and foundation, and approximate uniform torsion neglecting wall B. Values are for a unit load applied to the assembly. Average experimental slopes are given in Table 3 for comparison.

Comparison of theory and experiment shows that the best source of comparison is the value of F_D, the face wall shear. The rigid diaphragm solution, the first method shown above, is extremely poor while uniform torsion theory affords good correlation. The approximate uniform torsion theory solution, neglecting wall B, is reasonably good and much more practical than the more rigorous approach.

Studies of the two specimens, SD-6 and SD-7, showed the great influence of face walls on shear wall assemblies. The solution of box assemblies requires a complete re-evaluation of the theory. Modes of failure and general behavior are identical with those observed for individual wall and diaphragm elements.

TWO-STORY STRUCTURES WITHOUT FACE WALLS—SPECIMENS SD-8 and SD-9

In general, multi-story shear wall assemblies cannot be analyzed as a group of one-story structures. However, with all stories loaded, the errors involved may be small. This problem was studied with two specimens, SD-8 and SD-9, which were identical in structure but subjected to different loadings. The specimen details are shown in Fig. 22. Model SD-8 was loaded uniformly with equal loads in each panel and the roof diaphragm load half that of the second-story diaphragm. Specimen SD-9 was loaded in the same manner but in only one panel so that the structure was subjected to torsion.

Photographs of the specimens under test are shown in Figs. 23, 24 and 25. The specimens were constructed by precasting the walls and then assembling them in the testing jig at which time the concrete for the diaphragms was placed. All concrete was of the same mix and had a strength of 3,000 psi or higher at the time of test. The instrumentation was revised for these specimens. The horizontal loads, P, were applied by hydraulic jacks. Horizontal wall shears, R, were measured directly by calibrated hydraulic jacks that were continuously adjusted to keep the wall foundations in a straight line. Tie-down forces were measured directly with jacks and foundation rotations were eliminated.

Loads were applied in small increments after which all jacks were adjusted and forces and deflections measured. A continuous static equilibrium check was maintained to control errors.

Two series of tests were made on SD-8. The first run involved increasing the load up to first crack and then decreasing it back to zero. The second test was carried into the post ultimate range. Test results for Specimen SD-8 are given in Fig. 26. Data for SD-9 are given in Fig. 27. Deflection results for this model were inconsistent.

These structures are statically indeterminate to the second degree. Elastic analyses were made in which both bending and shear deflection of walls and diaphragms were included. With Specimen SD-8 the two-story computations yield,

$$R_A = R_C = 0.39P; \; R_B = 0.22P; \; T_A = T_C = 0.34p.$$

These theoretical relationships are plotted along with the experimental results. Note the excellent agreement. If single-story theory is used to solve the same problem, assuming a rigid diaphragm in addition, the following results are found:

$$R_A = R_C = 0.43P; \; R_B = 0.14P; \; T_A = T_C = 0.36P; \; T_B = 0.12P.$$

While the differences in R_B are large, other variations are small. Thus the error involved in using single-story theory in this particular problem is not excessive.

Similar computations were made for specimen SD-9. The theoretical results are shown in Table 3 for comparison and the two-story theory results are shown by dashed lines in Fig. 27.

Note that the results of the approximate single-story analysis are essentially identical to the more exact results. Some conclusions can be drawn if the reasons for this virtual identity are investigated.

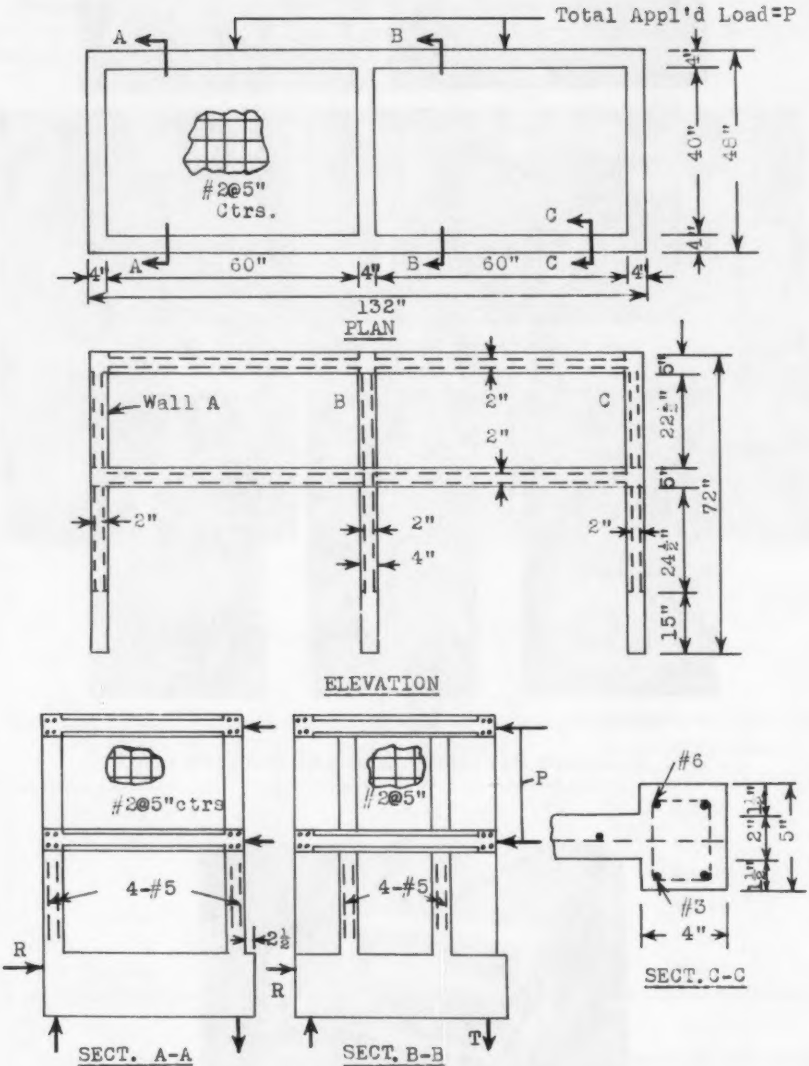
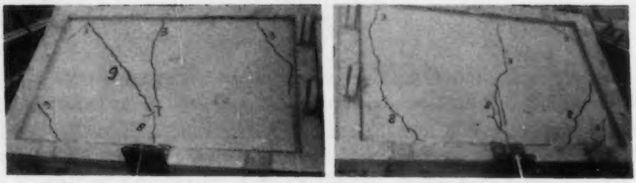
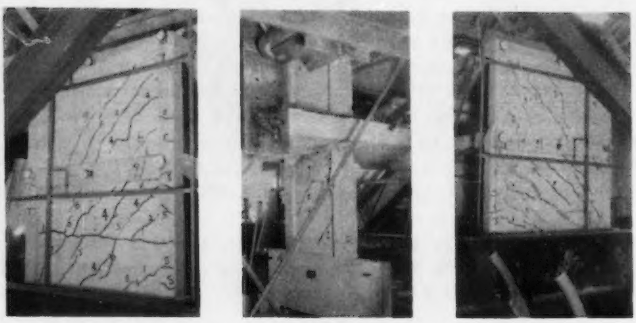


FIG. 22.—DETAILS OF SPECIMENS SD-8 AND SD-9.



Top Face of Upper Diaphragm

Bottom Face of Lower Diaphragm



Wall A Wall B Wall C

FIG. 23.—CRACK PATTERN FOR SD-8 AT MAXIMUM LOAD.

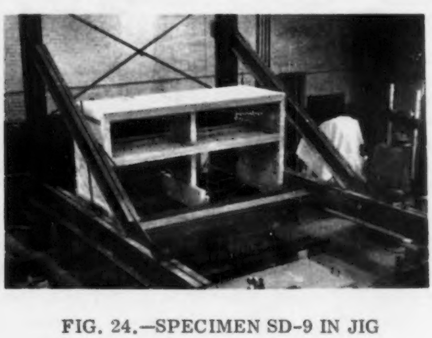
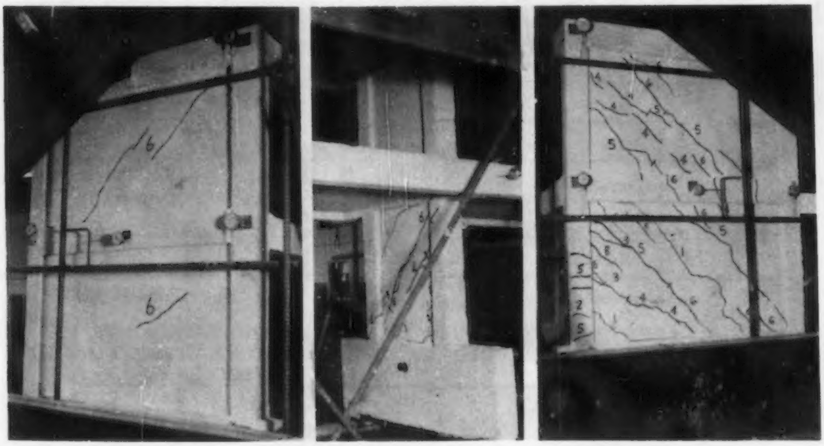


FIG. 24.—SPECIMEN SD-9 IN JIG

First, a major source of deformation is shearing distortion. If all distortion was from shear, the two theoretical computations would be identical providing that the diaphragm is rigid. This will be almost true if face walls are added.

Second, the loadings are fairly uniform. The two analyses differ greatly if only one story is loaded. Thus, practical loadings tend to minimize the theoretical differences.



Wall A Wall B Wall C

FIG. 25.—COMPONENT WALLS AT ULTIMATE.

TABLE 4

Force (1)	Two-Story Theory (2)	Single-Story Theory (3)
R_A	0.18P	0.18P
R_B	0.14P	0.14P
R_C	0.68P	0.68P
T_A	0.16P	0.15P
T_B	0.10P	0.12P
T_C	0.58P	0.57P

Finally, the structures are very rigid, making experimental errors large and the measured deflections of no real value in checking theory. With practical structures, items such as foundation distortion tend to invalidate complex analytical procedures. The best theoretical solution remains only a very rough estimate at best. If the entire SD series of specimens is studied as a whole, this is an inevitable conclusion.

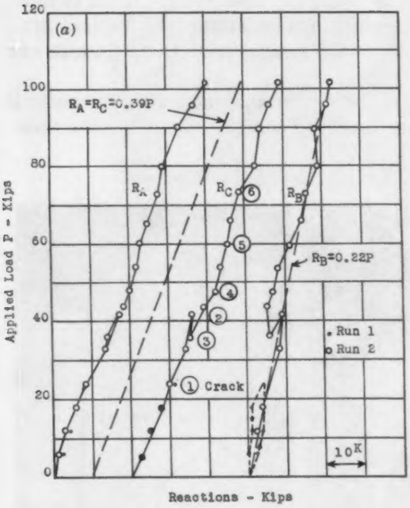


FIG. 26(a).—APPLIED LOAD VS. REACTIVE FORCES. SPECIMEN SD-8.

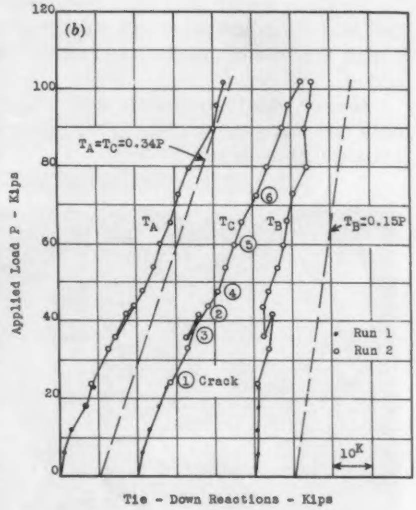


FIG. 26(b).—APPLIED LOAD VS. TIE-DOWN REACTIONS. SPECIMEN SD-8.

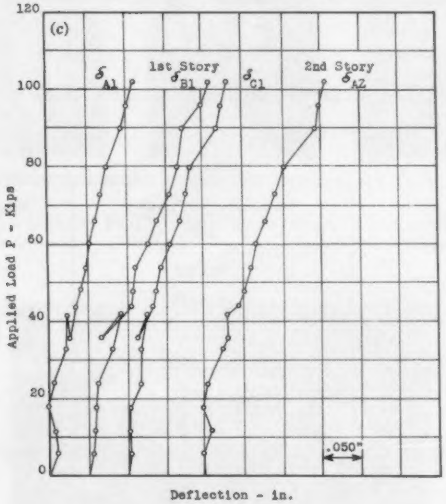


FIG. 26(c).—APPLIED LOAD VS. STORY DEFLECTION FOR SHEAR WALLS A, B, AND C. SPECIMEN SD-8.

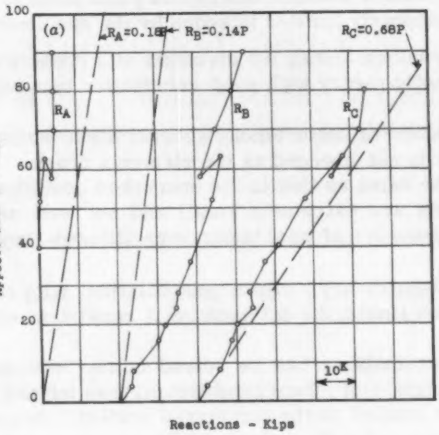


FIG. 27(a).—APPLIED LOAD VS. REACTIONS. SPECIMEN SD-9.

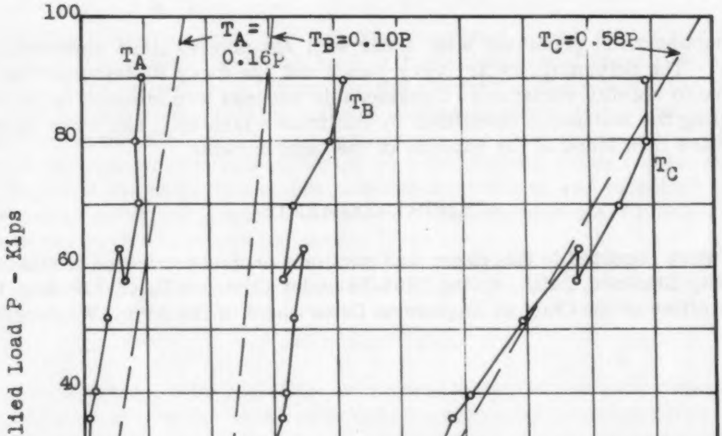


FIG. 27(b).—APPLIED LOAD VS. FOUNDATION TIE-DOWN FORCES. SPECIMEN SD-9.

CONCLUSIONS

The assembly studies made in this report yield several conclusions. These conclusions are obviously limited in scope by the specimens studied.

1. Force distribution among the elements of a shear wall assembly can be made by using elastic theory with good correlation between computed and observed results.
2. The distribution of shear among several shear walls that are in the uncracked condition is not changed as the elements crack. The distribution at ultimate load is the same as that in the uncracked condition.
3. Deformations are extremely small and not well predicted by theory. Foundation distortions are of vital importance although they may be very small in magnitude.
4. Uniform torsion theory yields a good solution. With closed sections providing the foundation limits, the deformation is consistent with the assumptions in the theory.
5. Two-story assemblies can be solved as two independent single-story structures using single-story theory and assuming an infinitely rigid diaphragm. This conclusion is limited to the structures studied. In general, if shearing distortion predominates in all walls and both structure and loading are regular, the structure can be analyzed using single-story theory. The errors in any one wall may be important but the overall analysis is reasonable and practical.

Instrumentation problems with shear wall assemblies limit experimental studies. The deformations are very small and the force distribution highly sensitive to rigidity variations. Considerable success was attained by closely controlling the foundation conditions by continuous jacking. The force distributions are then found at the expense of the deformations.

ACKNOWLEDGMENTS

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STRENGTH OF WELDED ALUMINUM COLUMNS

By R. J. Brungraber,[1] A. M. ASCE and J. W. Clark,[2] M. ASCE

SYNOPSIS

Welding heat-treated or cold-worked aluminum alloys causes partial annealing of the material in the vicinity of the welds so that the strength of the material near the welds is lower than the strength of the material in the rest of the structure. An experimental and analytical investigation was conducted to determine the effect of the varying mechanical properties on column strength.

The strength of aluminum columns with longitudinal welds can be computed by means of the same techniques used in determining the effect of residual stresses on column strength. Columns with transverse or other localized welds can be analyzed as stepped columns.

INTRODUCTION

Most aluminum alloys used for structural purposes are either heat treated or cold worked in order to develop higher mechanical properties than exist in the annealed temper. When alloys with enhanced properties are welded, the heat of welding reduces the strength in localized regions in and immediately adjacent to the welds. The purpose of this investigation was to determine the

Note.—Discussion open until January 1, 1961. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Structural Engineering Division, Proceedings of the American Society of Civil Engineers, Vol. 86, No. ST 8, August, 1960.

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strength of aluminum alloy columns containing regions of reduced strength resulting from welding. The investigation consisted of an analytical treatment, with experimental verification of some particular cases.

Notation.—The letter symbols adopted for use in this paper are defined where they first appear, in the illustrations or in the text, and are arranged alphabetically, for convenience of reference, in Appendix II.

ANALYSIS

Reduced-Strength Zone.—The mechanical properties in the vicinity of a weld in an aluminum alloy vary from a minimum at or near the center of the weld to unaffected parent metal properties at some distance away from the weld. As an example, Fig. 1 shows a typical distribution of yield-strength in the vicinity of a weld.

Although a completely rigorous analysis of a welded column should take into account the continuous variation of the stress-strain characteristics in the vicinity of a weld, a satisfactory solution can be obtained by considering that in the vicinity of each weld there is a "reduced-strength zone" in which the stress-strain curve is that of the softest material in or near the weld, while the stress-strain curve outside such zones is that of unaffected parent metal. This reduced-strength zone does not cover the entire region in which there is any reduction in strength due to the heat of welding but rather covers a region such that, if the material in the reduced-strength zone is considered to have the mechanical properties of the most highly heat-affected material adjacent to the weld while the rest of the material is considered to have the mechanical properties of unaffected parent metal, the resulting weighted average properties will satisfactorily represent the behavior of the actual welded member.

An approximate value of the extent of the reduced-strength zone can be obtained by integrating the area under a curve representing the distribution of tensile strength, yield strength, or hardness in the vicinity of welds.[3] For example, if tensile strength is used, the extent of the reduced-strength zone must be such that the sum of the reduced-strength area times the minimum tensile strength plus the area outside the reduced-strength zones times the parent-metal tensile strength equals the integrated tensile strength. Theoretically, the extent of the reduced-strength zone might differ, depending on whether it is determined on the basis of tensile strength, yield strength, or hardness. Practically, however, all these methods of establishing the reduced-strength zone give approximately the same values. The present paper demonstrates that values so determined are satisfactory for use in computing the strength of welded columns.

Columns Containing Only Longitudinal Welds.—In analyzing the effect of welds on buckling strength, it is convenient to divide welds into two categories; (1) longitudinal welds that affect an appreciable proportion of the length, and (2) transverse or localized welds, that affect only a small proportion of the length. Examples of the first type are continuous longitudinal welds used to build up large members from plates or rolled shapes. Examples of the second type are welds used for splices or to attach bracing members to the column.

Columns containing only longitudinal welds, extending essentially the full length of the column, can be analyzed in the same manner as columns having

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3 "Design of Welded Aluminum Structures," by H. N. Hill, J. W. Clark, and R. J. Brungraber, Proc. 2528, June, ST.

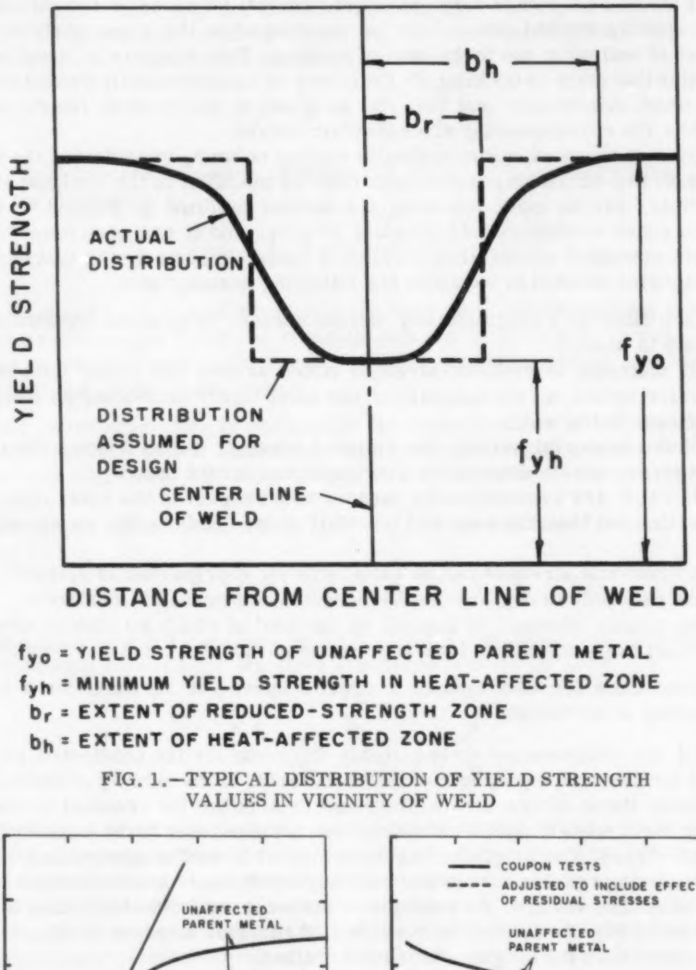

FIG. 1.—TYPICAL DISTRIBUTION OF YIELD STRENGTH VALUES IN VICINITY OF WELD

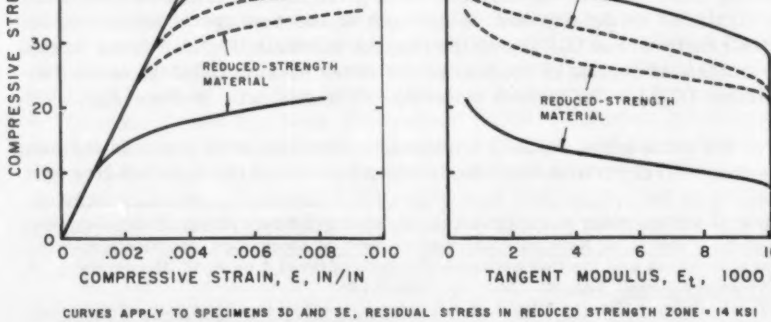

FIG. 2.—COMPRESSIVE STRESS-STRAIN AND STRESS-TANGENT MODULUS CURVES FOR LONGITUDINALLY WELDED 6061-T6 COLUMNS

residual stresses.[4,5] In fact, the effect of residual stresses on the strength of a longitudinally welded column can be considered in the same analysis with the effect of softening due to the heat of welding. This analysis is based on the assumption that prior to buckling all the fibers in a longitudinally welded column are strained indentically and that the stresses in the various fibers can be found from the corresponding stress-strain curves.

An exact analysis of a longitudinally welded column, considering the compressive stress-strain diagram of each fiber as modified by the residual stress in that fiber, can be made following the method outlined by Osgood.[4] However, the exact analysis would be quite lengthy, and it has been found that a simplified method of computation results in close agreement with test results. This simplified method is based on the following assumptions:

1. Each fiber of a longitudinally welded column is strained identically as the column is loaded.
2. All material in reduced-strength zones around the welds follows the compressive stress-strain diagram of the most highly heat-affected material in or adjacent to the welds.
3. All the material outside the reduced-strength zones follows the compressive stress-strain diagram of the unaffected parent metal.
4. All welds are symmetrically located with respect to the centroidal axis of the section (so that the axis will not shift under loads in the plastic stress range).
5. Any residual stresses can be satisfactorily represented as uniform tension in the reduced-strength zones and uniform compression elsewhere.
6. The column strength is limited by the load at which an ideally straight column could begin to deflect laterally ("tangent modulus" critical load).[6]

The procedure for establishing a column curve for columns for a given cross section is as follows:

1. Plot the compressive stress-strain diagrams for the unaffected parent metal and for the most highly heat-affected material in the vicinity of the welds.
2. Modify these stress-strain diagrams to account for residual stresses by shifting the origins to points on the curves corresponding to the initial values of residual stress. The origin for the parent metal is shifted upward to account for residual compressive stress and the origin for the reduced-strength material is shifted downward. An example of such curves for welded alloy 6061-T6 is shown in Fig. 2. To picture the effect of residual stresses in Fig. 2, the curves, rather than the origins, have been shifted.
3. Using these curves, the stiffness for a given column cross-section under a given strain (ϵ) is determined as the sum of the moment of inertia of the reduced-strength areas (I_r) times the tangent modulus (E_h) for these areas, plus the moment of inertia of the rest of the cross section ($I-I_r$) times the tangent modulus (E_o) for the parent material. The stiffness is then $[E_h I_r + E_o (I-I_r)]$.
4. For the same given strain (ϵ), the load on the column is found as the sum of the reduced-strength area (A_r) times the stress from the reduced-strength,

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[4] "The Effect of Residual Stresses on Column Strength," by William R. Osgood, Proceedings of the First U. S. National Congress of Applied Mechanics, pp. 415-418.

[5] "Residual Stress and the Compressive Strength of Steel," by A. W. Huber and L. S. Beedle, Welding Journal, Vol. 33, 1954, pp. 589s-614s.

[6] "Inelastic Column Theory," by F. R. Shanley, Journal of the Aeronautical Sciences, Vol. 14, 1947, pp. 261-268.

stress-strain curve (σ_h) for the given strain plus the remainder of the area ($A-A_r$) times the parent metal stress (σ_o) for the same strain. The average stress (σ_c) corresponding to the buckling load is

$$\sigma_c = \frac{A_r}{A}\sigma_h + \left(1 - \frac{A_r}{A}\right)\sigma_o \quad \text{(1)}$$

5. Compute the effective slenderness ratio ($K\,L/r$) corresponding to σ_c from the formula

$$\frac{K\,L}{r} = \pi\sqrt{\frac{E_h\,I_r/I + E_o\left(1 - I_r/I\right)}{\sigma_c}} \quad \text{(2)}$$

The values of I_r, I, r, and K refer to the axis about which buckling occurs.

By repeating the foregoing computations for a number of values of strain, a plot of slenderness ratio versus buckling strength can be established for columns of a given cross section.

The above procedure is essentially the same as that outlined by Huber and Beadle[5] with the exception that for aluminum alloys the compressive modulus never reaches zero, and thus all portions of the cross section must be considered.

For cross sections for which $I_r/I = A_r/A$, the stiffness becomes

$$E_h\,I_r + E_o\left(I - I_r\right) = I\left[\frac{A_r}{A}E_h + \left(1 - \frac{A_r}{A}\right)E_o\right] \quad \text{(3)}$$

Substituting the stiffness given by Eq. 3 in Eq. 2 and combining with Eq. 1 gives the following formula for effective slenderness ratio ($K\,L/r$):

$$\frac{K\,L}{r} = \pi\sqrt{\frac{\frac{A_r}{A}E_h + \left(1 - \frac{A_r}{A}\right)E_o}{\frac{A_r}{A}\sigma_h + \left(1 - \frac{A_r}{A}\right)\sigma_o}} \quad \text{(4)}$$

Examples of column buckling curves based on Eqs. 1 and 4 for some typical welded aluminum alloys are shown in Fig. 3, 4, and 5. Test points for some actual columns are also shown on these figures.

Although the location of a weld in the cross section affects the column strength, it would be inconvenient to consider this effect in design since it would be necessary to have curves of stress versus length or slenderness ratio for various values of A_r/A for a large variety of cross sections. It is believed that the column curves based on Eq. 4 (that involves the assumption that $I_r/I = A_r/A$) can be used satisfactorily for design purposes, provided that care is exercised in avoiding large concentrations of welding at the extreme fibers of the cross section. Further investigation of this feature of design is planned.

Columns Containing Only Transverse Welds.—Columns containing transverse welds can be considered as stepped columns, with the reduced-strength zones of the transverse welds representing short lengths of the column having reduced stiffness. Instead of the moment of inertia varying as in the typical stepped column, the modulus varies. The effect in either case is to vary the stiffness, E I.

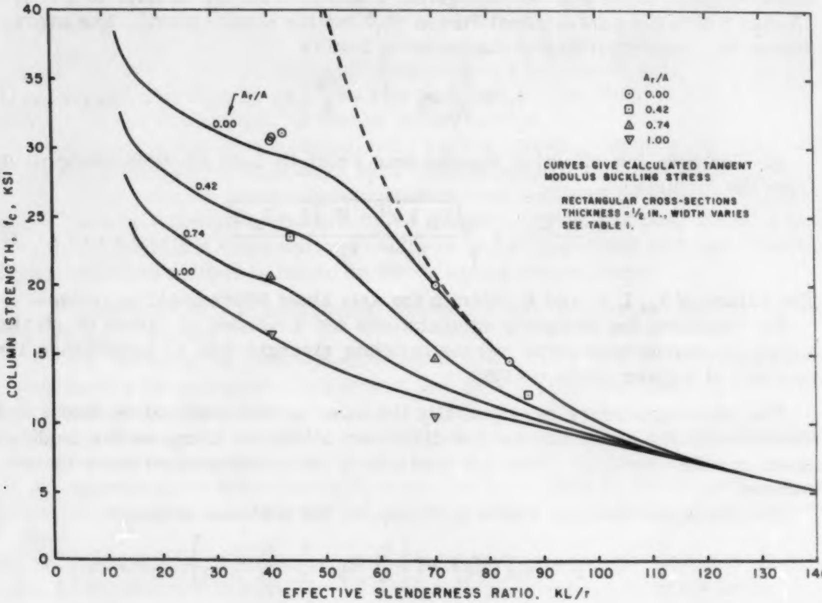

FIG. 3.—BUCKLING STRENGTH OF LONGITUDINALLY WELDED COLUMNS OF 5456-H321

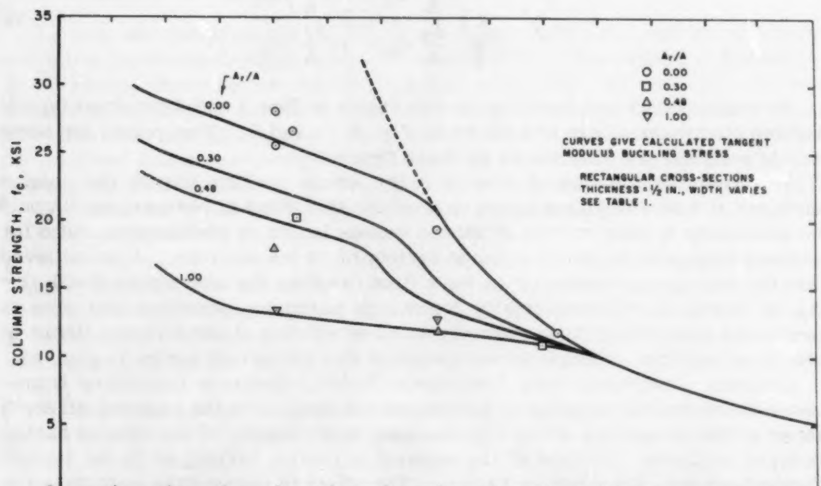

FIG. 4.—BUCKLING STRENGTH OF LONGITUDINALLY WELDED COLUMNS OF 5154-H34

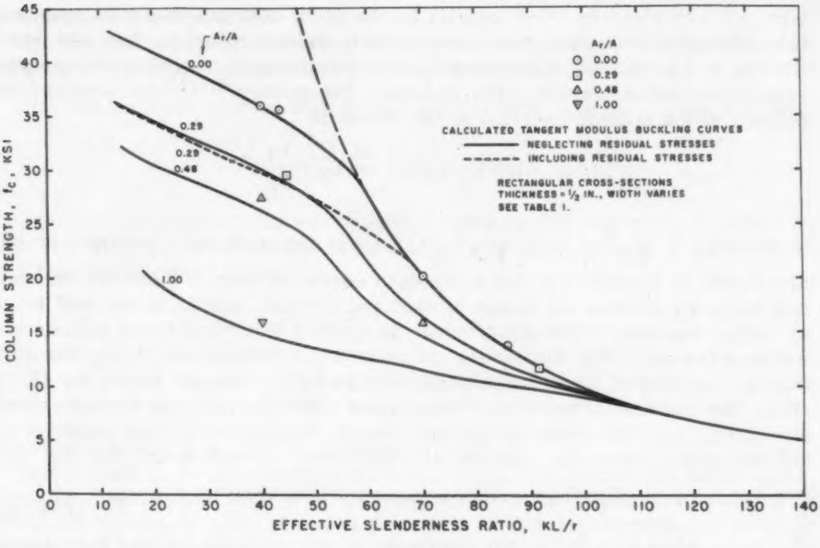

FIG. 5.—BUCKLING STRENGTH OF LONGITUDINALLY WELDED COLUMNS OF 6061-T6

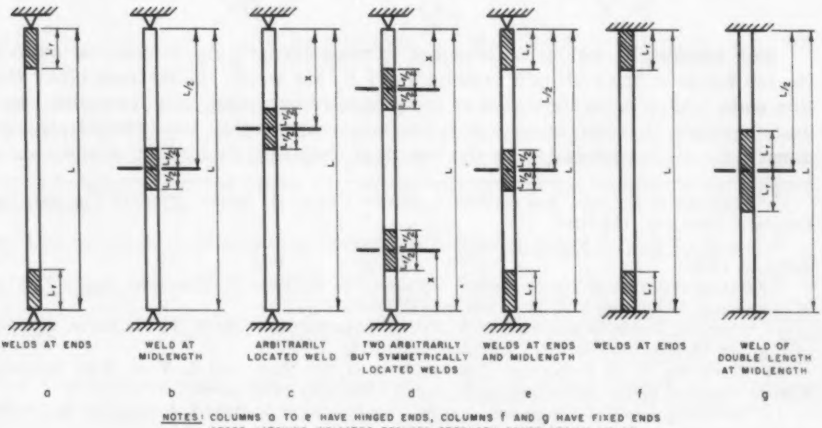

FIG. 6.—TRANSVERSELY WELDED COLUMNS CONSIDERED IN ANALYTICAL TREATMENT

Many solutions for particular stepped columns appear in the literature,[7,8,9,10,11,12,13,14,15] usually in the form of transcendental equations. In applying these equations to transversely welded columns, they can be rewritten in a form that is convenient for determining curves of buckling stress versus slenderness ratio. For example, the solution[8,9,10] for a pinned-end column with a stiffened section at the center is:

$$\tan k_1 L_1 \tan k_2 L_2 = \frac{k_1}{k_2} \quad \ldots \ldots \ldots \ldots \ldots \quad (5)$$

in which $k_1 = \sqrt{\frac{P}{(E\,I)_1}}$, $k_2 = \sqrt{\frac{P}{(E\,I)_2}}$, $(E\,I)_1$ is the stiffness of sections at ends of column, $(E\,I)_2$ refers to the stiffness of center section, P indicates the buckling load, L_1 denotes the length of each end section, and L_2 is the half length of center section. This solution can be applied to a pinned-end column with welds at the ends (Fig. 6(a)) using the following substitutions: $(E\,I)_o$, the stiffness of sections of the column outside the reduced-strength zones, for $(E\,I)_2$; $(E\,I)_h$, the stiffness of sections of the column within the reduced-strength zones, for $(E\,I)_1$; L_r, the reduced-strength length for each weld (the length of the column contained in the reduced-strength zone of each weld), for L_1; $L_2 = \frac{L}{2} = L_r$; L for the length of welded column with buckling load, P; $k_1 = \sqrt{\frac{P}{(E\,I)_h}} = \frac{\pi}{L_h}$; $k_2 = \sqrt{\frac{P}{(E\,I)_o}} = \frac{\pi}{L_o}$; L_o for the length of prismatic unaffected parent metal column with buckling load, P; L_h for the length of prismatic column of reduced-strength material with buckling load, P; and introducing r, the radius of gyration common to all the above columns. The solution becomes

$$\frac{L}{r} = \frac{L_o}{r} - \left(\frac{2\,L_o}{\pi\,r} \tan^{-1}\left[\frac{L_h/r}{L_o/r} \tan\left(\frac{\pi\,L_r/r}{L_h/r}\right)\right] - \frac{2\,L_r}{r}\right) \quad \ldots \ldots \quad (6)$$

For transverse welds, the reduced-strength length, L_r, is considered to be b_r for welds at the ends of a column and 2 b_r for welds at locations other than the ends, where b_r is the extent of reduced-strength zone. For localized longitudinal welds, that may be considered as transverse welds, the reduced-strength length, L_r, is considered to be the length of the weld, plus b_r for welds located

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7 "Buckling of Uniform and Stepped Columns," by J. H. Meier, Product Engineering, October, 1949, pp. 119-123.

8 Theory of Elastic Stability, by S. Timoshenko, McGraw-Hill Book Company, First Edition, 1936.

9 "Matrix Solution of the N-Section Column," by William T. Thomson, Journal of the Aeronautical Sciences, Vol. 16, 1949, pp. 623-624.

10 "Critical Loads of Columns of Varying Cross Section," by W. T. Thomson, Journal of Applied Mechanics, Vol. 17, 1950, pp. 132-134.

11 "Buckling of an N-Section Column," by G. Sri Ram and G. V. R. Rao, Readers Forum, Journal of the Aeronautical Sciences, Vol. 19, 1952, p. 66.

12 "Buckling Load of a Stepped Column," by Frederic M. Hoblit, Journal of the Aeronautical Sciences, Vol. 18, 1951, pp. 124-138.

13 "Critical Loads on Variable-Section Columns in the Elastic Range," by Gordon Silver, Journal of Applied Mechanics, Vol. 18, 1951, pp. 414-420.

14 "Buckling Loads for Columns of Variable Cross Section," by B. E. Gatewood, Readers Forum, Journal of the Aeronautical Sciences, Vol. 21, 1954, pp. 287-288.

15 "Analysis of Nonuniform Columns and Beams by a Simple D. C. Network Analyzer," by George W. Swenson, Jr., Journal of the Aeronautical Sciences, Vol. 19, 1952, pp. 273-278.

at the ends of a column and the length of the weld, plus 2 b_r for welds at locations other than the ends.

In a similar manner, other solutions for stepped columns can be applied to transversely welded columns, giving the following results:

For a pinned-end column with a weld at midlength (Fig. 6(b)), Eq. 5 yields

$$\frac{L}{r} = \frac{L_r}{r} + \frac{2\ L_o}{\pi\ r} \tan^{-1}\left[\frac{L_h/r}{L_o/r} \cot\left(\frac{\pi\ L_r/r}{2\ L_h/r}\right)\right] \quad \ldots\ldots\ldots\ldots \quad (7)$$

For a pinned-end column with a weld at an arbitrary distance, X, from one end (Fig. 6(c)), the solution for a three-section stepped column of Thomson,[10] or Thomson's[9] case I gives

$$\frac{L}{r} = \frac{L_o}{r} - \left(\frac{L_o}{\pi\ r} \tan^{-1}\left\{\frac{L_h/r}{L_o/r} \tan\left[\frac{\pi\ L_r/r}{L_h/r} + \tan^{-1}\left(\frac{L_o/r}{L_h/r} \tan \frac{\pi\left(X - L_r/2\right)/r}{L_o/r}\right)\right]\right\} - \frac{L_r}{2\ r} - \frac{X}{r}\right) \quad \ldots\ldots \quad (8)$$

By substitution of X = L/2, Eq. 8 reduces to Eq. 7.

Eq. 8 establishes the slenderness ratio, L/r, of a pinned-end column containing a single transverse weld as being equal to the slenderness ratio of an unwelded column, L_o/r, minus a factor that reflects the reduced-strength length, L_r, and the location, X, of the weld. This suggests the possibility of treating columns containing two or more transverse welds by the subtraction of a series of factors from L_o/r (one factor for each weld). By means of geometrical considerations of the kind used by Meier,[7] it can be demonstrated that this method of computation is satisfactory, provided that the value of X for each transverse weld is replaced by the quantity X'_n given by the formula:

$$X'_n = X_n + r \sum_{m=1}^{n-1} C_m \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

in which X'_n is the value of X' for nth weld from end of column, X_n refers to the distance from center of nth weld to end of column, and C_m indicates the length correction factor for mth weld from end of column ($m < n$). The value of the length correction factor C_n for the nth weld from the end of the column is given by

$$C_n = \frac{L_o}{\pi\ r} \tan^{-1}\left\{\frac{L_h/r}{L_o/r} \tan\left[\frac{\pi\ L_r/r}{L_h/r} + \tan^{-1}\left(\frac{L_o/r}{L_h/r} \tan \frac{\pi\left(X'_n - L_r/2\right)/r}{L_o/r}\right)\right]\right\} - \frac{L_r}{2\ r} - \frac{X'_n}{r} \quad \ldots\ldots \quad (10)$$

The slenderness ratio of a column with a total of N transverse or localized welds is

$$\frac{L}{r} = \frac{L_o}{r} - \sum_{n=1}^{N} C_n \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

The values of C_n may be found for each weld successively, starting from one end of the column. The solution can be simplified somewhat by starting from both ends of the column and working toward the center.

Eq. 11 agrees with the particular cases solved in by Thomson.[9,10] For example, for a pinned-end column containing two welds of reduced-strength length, L_r, located equal distances, X, from the two ends (Fig. 6(d)), the solution can be obtained from Eq. 11 as

$$\frac{L}{r} = \frac{L_o}{r} - \left(\frac{2\ L_o}{\pi\ r} \tan^{-1}\left\{\frac{L_h/r}{L_o/r} \tan\left[\frac{\pi\ L_r/r}{L_h/r} + \tan^{-1}\left(\frac{L_o/r}{L_h/r} \tan \frac{\pi\left(X - L_r/2\right)/r}{L_o/r}\right)\right]\right\} - \frac{L_r}{r} - \frac{2X}{r}\right) \ldots\ldots (12)$$

For the case of a pinned-end column containing welds at the ends and at the center (Fig. 6(e)), the solution can be obtained from Eq. 11 as

$$\frac{L}{r} = \frac{L_o}{r} - \frac{2\ L_o}{\pi\ r}\left\{\tan^{-1}\left[\frac{L_h/r}{L_o/r} \tan \frac{\pi\ L_r/r}{L_h/r}\right] - \tan^{-1}\left[\frac{L_h/r}{L_o/r} \cot \frac{\pi\ L_r/r}{2\ L_h/r}\right] + \frac{\pi}{2} - \frac{3\ \pi\ L_r/r}{2\ L_o/r}\right\} \ldots\ldots\ldots (13)$$

By proper substitution, Eqs. 12 and 13 can be shown to agree with the solution for a three-section stepped column, such as Thomson's case 3.[9,11]

The foregoing equations for pinned-end columns are also valid for fixed-end columns having transverse welds so placed that the fixed-end column can be considered as a pinned-end column of half the length. This is correct, however, only for fixed-end columns with welds located symmetrically about the quarter points. For example, the solution for a pinned-end column with welds at the ends is also valid for a fixed-end column having welds at the quarter points, and the solution for a pinned-end column with a weld at midlength is valid for a fixed-end column with welds at the ends and the middle, the reduced-strength zone for the weld at the middle covering twice as much column length as does that for each of the two end welds.

For the case of a fixed-end column having welds at the ends (Fig. 6(f)), no solution for a pinned-end column applies directly, however, by combining the solution for a pinned-end column containing no welds and one containing a weld at the middle, the solution for a fixed-end column with a reduced-strength zone of length L_r at each end is

$$\frac{L}{r} = \frac{L_o}{r} + \frac{2\ L_r}{r} + \frac{2\ L_o}{\pi\ r} \tan^{-1}\left[\frac{L_h/r}{L_o/r} \cot\left(\frac{\pi\ L_r/r}{L_h/r}\right)\right] \ldots\ldots\ldots (14)$$

in which L is the length of the fixed-end column and L_o and L_h are the lengths of pinned-end columns with the same buckling stress. Eq. 14 is also valid for a fixed-end column with a reduced-strength zone of length 2 L_r at the middle (Fig. 6(g)). For fixed-end columns with welds that are not symmetrical about the points of inflection or for columns with partial restraint at the ends, the foregoing formulas can be used to obtain approximate solutions, even though shifting of the points of inflection during loading introduces some error in the computations. In these cases, distances from the ends of the column in the formulas for pinned-end columns are replaced by distances from the initial points of contraflexure, and values of L, L_o, and L_h are replaced by K L, K L_o, and K L_h.

For transverse welds that do not affect the entire cross section but are still symmetrically placed with respect to the axis of the column, the foregoing

equations can be used by replacing the slenderness ratio for an all-reduced-strength column by that for a longitudinally welded column with the same reduced-strength zones as the transversely welded cross section of the column in question. Comparison of some stepped column analyses with test results are shown in Figs. 7, 8, and 9.

The formulas for transversely welded columns developed in this section take the general form of Eq. 11. To use such formulas directly in design, the designer could select a trial column section, determine the average stress on the trial section, compute a value of L/r from the appropriate formula (Eqs. 6 to 14), and compare the computed L/r with the actual slenderness ratio of the column. If the actual slenderness ratio were less than the computed value, the trial section would carry the load safely. This procedure has the disadvantage that it does not give an allowable column stress directly. For design purposes, therefore, it may be more practical to use the stepped column analysis to develop approximate methods for obtaining allowable stresses directly. For example, it can be demonstrated that transverse welds whose reduced-strength zones do not extend more than 0.05 L from the end of a pinned-end column have a negligible effect on buckling strength. For welds outside these regions, curves can be established giving strength-reduction factors that depend on the locations and sizes of the welds and the proportions of the columns. It is convenient to put such curves in a form so that they yield equivalent values of A_r/A to be used with buckling curves for longitudinally welded columns.

Columns Containing Both Longitudinal and Transverse Welds.—A column containing both transverse and longitudinal welds can be analyzed as a column containing only transverse welds by replacing the curve of slenderness ratio versus stress for an all parent-metal column with the curve for a column similar to the one in question but containing only the longitudinal welds. Such a curve can be computed by the method of columns containing only longitudinal welds.

DESCRIPTION OF TEST SPECIMENS

The longitudinally welded column specimens were made from 1/2-in. thick plate of alloys 6061-T6, 5154-H34, and 5456-H321. Each specimen consisted of two plates joined by a butt weld to form a solid rectangular section. For such a section, $I_r/I = A_r/A$. The transversely welded column specimens were made from 1/4-in. and 1/2-in. thick plate and 3-in. OD by 1/4-in. wall thickness tube—all of alloy 6061-T6. Average mechanical properties are listed in Table 4. All the specimens made from plate were welded by the argon-shielded, consumable electrode process, using parent metal filler for the 5456-H321 and the 5154-H34 plates and 4043 filler for the 6061-T6 plates. The tubular specimens were welded by the argon-shielded, tungsten electrode process, using 5556 filler wire. After being welded, the plate specimens were sawed to size and then straightened; the transversely welded specimens by plastic bending and the longitudinally welded specimens by machining. In straightening the transversely welded column specimens, it was necessary to bend one end relative to the other about 3°. All the bending was confined to the reduced-strength zone immediately adjacent to the weld bead.

After straightening, the specimens were finish-machined with parallel edges and ends flat, parallel, and perpendicular to the axis of the column. To determine whether bending the transversely welded columns, in order to straighten them, introduced sufficient strain hardening or residual stress to affect the

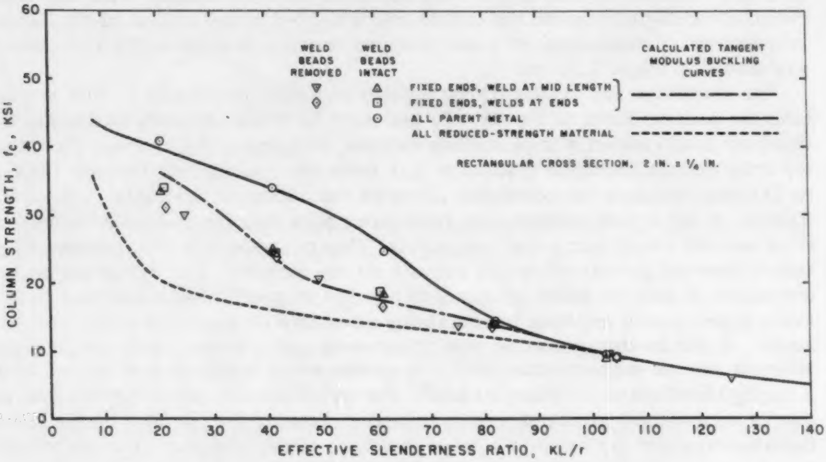


FIG. 7.—BUCKLING STRENGTH OF TRANSVERSELY WELDED COLUMNS OF 6061-T6

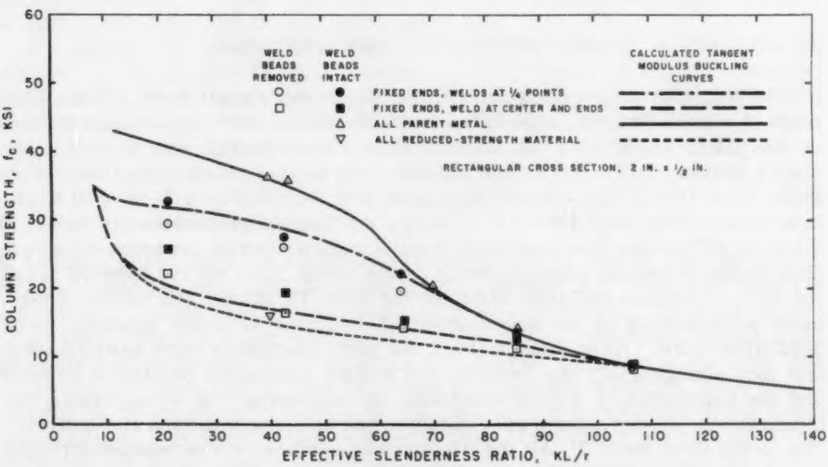


FIG. 8.—BUCKLING STRENGTH OF TRANSVERSELY WELDED COLUMNS OF 6061-T6

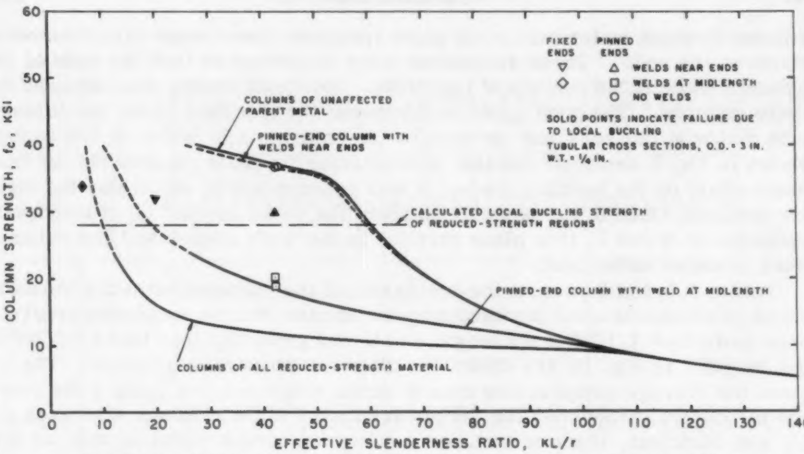
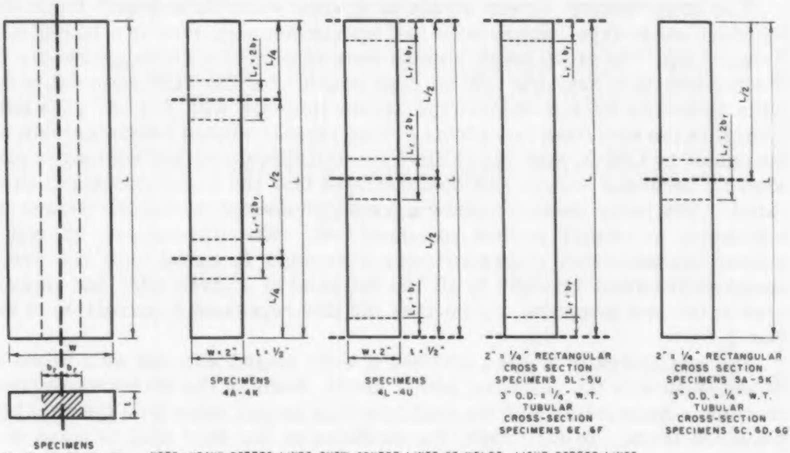
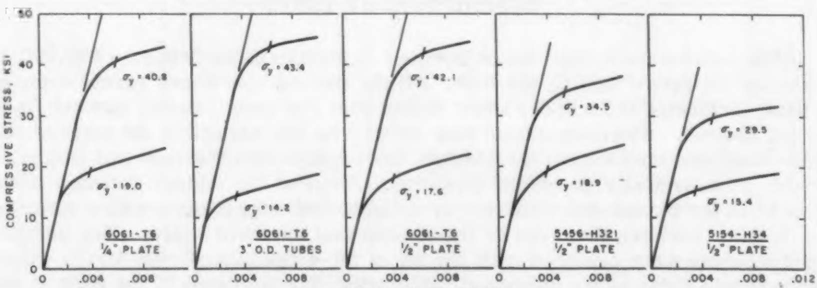


FIG. 9.—STRENGTH OF TRANSVERSELY WELDED TUBULAR COLUMNS OF 6061-T6



NOTE: HEAVY DOTTED LINES SHOW CENTER LINES OF WELDS, LIGHT DOTTED LINES AND CROSS-HATCHING SHOW REDUCED-STRENGTH REGIONS

FIG. 10.—DETAILS OF WELDED COLUMN SPECIMENS



NOTE: LOWER CURVES ARE FOR MATERIAL IN REDUCED-STRENGTH ZONE, UPPER CURVES ARE FOR UNAFFECTED PARENT METAL

FIG. 11.—COMPRESSIVE STRESS-STRAIN DIAGRAMS REPRESENTIVE OF THE MATERIAL USED IN THE WELDED COLUMN SPECIMENS

column strength materially, some plate specimens were made with transverse welds at the ends. These specimens were machined so that the ends of the columns were at the centers of the welds. No straightening was required for these columns. The good agreement between the test results for the columns with welds at the ends and the results for columns with welds at the center, shown in Fig. 7, demonstrates that straightening the latter columns did not have much effect on the buckling loads. It was unnecessary to straighten the tubular columns since it was possible to place the small amount of crookedness, recorded in Table 3, in a plane parallel to the knife edges, and the columns were centered under load.

Tables 1, 2, and 3 present the dimensions of the columns, including the measured crookedness after straightening. In no case was the measured crookedness more than 1/1000 of the length, and it was generally less than 1/10,000 of the length. In Fig. 10 are shown the different kinds of specimens. Fig. 11 gives the average compressive stress-strain diagrams, and Table 4 the average mechanical properties for the material used in the columns. For each alloy and thickness, there is included a curve for parent metal as well as one for the most highly heat-affected material adjacent to the weld.

The compressive stress-strain diagrams were determined from ASTM standard sheet-type compression test specimens supported in a Montgomery-Templin jig.[16] In most cases, strains were measured with Huggenberger Type F extensometers having a 1/2-in. gage length. For the 5154 material, strains were indicated by a Templin-type strain follower with a 1-in. gage length. Strains in the specimen taken from a transversely welded tubular column were measured by 1/4-in. gage length SR-4 electrical resistance strain gages placed where a hardness survey had demonstrated that the softest material was located. The latter measurements were supplemented by similar strain measurements on tubular column specimen 6-G. Where necessary, the experimental stress-strain diagrams were corrected to agree with the average compressive yield strength of all the material of a given alloy and thickness used in the test program. In no case did this represent a correction of more than 3.4%.

The longitudinally welded specimens were tested with the weld beads machined flush with the adjoining parent metal. Some of the transversely welded specimens were tested with the weld beads intact and some with the weld beads machined flush. In each case, the condition of the weld bead is noted in the tables and figures of test results.

DESCRIPTION OF TESTS

The column tests were made in either a 50,000-lb Baldwin or a 300,000-lb Amsler Universal testing machine. All the column specimens except some of those fabricated from tubes were tested with flat ends bearing against fixed steel platens. Previous experience as well as the results of the tests of the unwelded specimens from the present investigation demonstrate that this provides an essentially fixed-end condition. Some of the tubular columns were tested in the pinned-end condition by using knifed-edge platens with a distance of 1.25 in. between the faces of the platens and the knife edges. The pinned-end columns were centered with the aid of SR-4 resistance-type strain gages on opposite sides of the column at midlength. The positions of the ends of the

16 Tentative Methods of Compression Testing of Metallic Materials, Anon., ASTM Designation E9-52T.

TABLE 1.—PROPERTIES OF LONGITUDINALLY WELDED COLUMNS

Specimen No.	Length, L, in.	Width, b, in.	Thickness, t, in.	Extent of Reduced-Strength Zone, b_r, in.	Ratio of Reduced-Strength Area to Total Area[a] $A_r/A = 2b_rt/bt$	Effective Slenderness Ratio,[b] KL/r	Failure Load, P kips	Column Strength, f_c, ksi: Measured ($f_c = P/A$)	Column Strength, f_c, ksi: Computed	Computed / Measured f_c	Maximum Crookedness,[d] in.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
					5154-H34 Welded with 5154						
1-A	11.60	0.750	0.502	0.60	1.00	40.0	5.05	13.4	13.1	0.98	0.0004
1-B	14.82	0.750	0.367	0.60	1.00	69.9	3.82	12.8	11.9	0.93	0.0016
1-C	8.80	2.50	0.383	0.60	0.48	39.6	17.19	18.0	20.1	1.12	0.0005
1-D	15.62	2.50	0.387	0.60	0.48	69.7	11.55	12.0	13.7	1.14	0.0026
1-E	12.25	4.00	0.483	0.60	0.30	43.9	39.30	20.3	21.8	1.08	0.0050
1-F	23.98	4.00	0.464	0.60	0.30	89.5	20.30	10.9	11.6	1.07	0.0020
1-G	10.60	1.00	0.458	0	0.00	40.1	12.88	28.1	26.0	0.93	0.0002
1-H	9.69	2.50	0.420	0	0.00	40.0	26.96	25.6	26.0	1.02	0.0010
1-J	16.39	2.50	0.405	0	0.00	70.0	19.85	19.4	20.7	1.07	0.0020
1-K	12.25	4.00	0.493	0	0.00	43.1	52.65	26.7	25.7	0.96	0.0030
1-L	23.98	4.00	0.450	0	0.00	92.2	21.40	11.9	11.9	1.00	0.0020
					5456-H321 Welded with 5456						
2-A	11.81	0.751	0.510	0.74	1.00	40.2	6.28	16.4	16.3	0.99	0.0004
2-B	14.68	0.749	0.363	0.74	1.00	69.9	2.85	10.5	11.7	1.11	0.0010
2-C	9.20	2.00	0.398	0.74	0.74	40.0	16.46	20.7	20.3	0.98	0.0004
2-D	12.82	2.00	0.318	0.74	0.74	69.8	9.45	14.8	12.9	0.87	0.0039
2-E	12.25	3.50	0.489	0.74	0.42	43.5	40.40	23.6	24.1	0.98	0.0030
2-F	23.98	3.50	0.476	0.74	0.42	87.2	20.20	12.1	11.5	0.95	0.0002
2-G	10.70	0.750	0.464	0	0.00	39.9	10.65	30.7	29.8	0.97	0.0036
2-H	18.83	0.751	0.465	0	0.00	70.3	7.00	20.1	20.0	0.99	0.0021
2-J	11.39	2.00	0.492	0	0.00	40.1	30.40	30.9	29.7	0.96	0.0005
2-K	12.25	3.50	0.501	0	0.00	42.2	54.60	31.2	29.5	0.95	0.0030
2-L	23.98	3.50	0.495	0	0.00	83.9	25.35	14.6	14.5	0.99	0.0020
					6061-T6 Welded with 4043						
3-A	10.20	1.00	0.440	0.72	1.00	40.1	7.02	16.0	14.8	0.93	0.0009
3-B	10.39	3.00	0.452	0.72	0.48	40.0	37.36	27.6	26.3	0.95	0.0006
3-C	15.50	3.00	0.383	0.72	0.48	69.8	18.45	16.1	14.8	0.92	0.0008
3-D	12.25	5.00	0.475	0.72	0.29	44.7	70.20	29.6	28.8[c]	0.97	0.0050
3-E	23.98	5.00	0.455	0.72	0.29	91.5	27.00	11.9	11.9[c]	1.00	0.0001
3-F	10.20	0.750	0.442	0	0.00	39.8	11.98	36.1	36.2	1.00	0.0006
3-G	17.50	3.00	0.433	0	0.00	70.0	26.40	20.4	20.3	0.99	0.0035
3-H	12.25	5.00	0.488	0	0.00	43.5	87.20	35.8	35.0	0.98	0.0050
3-J	23.98	5.00	0.484	0	0.00	85.6	33.70	13.9	13.5	0.97	0.0001

[a] For cases where $2b_r$ is greater than b the value of A_r/A is taken as 1.00.
[b] K is considered as 0.5 in all cases; r = 0.289 t, in.
[c] Including Effect of Residual Stresses.
[d] Maximum deviation of axis of column from straight line through ends.

TABLE 2.—PROPERTIES OF TRANSVERSELY

Specimen No. (1)	Location of Welds (2)	Condition of Weld Beads (3)	Length, L, in. (4)	Width, b, in. (5)	Thickness, t, in. (6)	Radius of Gyration, r, in. (7)
4-A	1/4 points	Intact	6.18	2.00	0.490	0.141
4-B			12.02	2.00	0.489	0.141
4-C			18.01	2.00	0.488	0.141
4-D			24.03	2.00	0.489	0.141
4-E			29.99	2.00	0.489	0.141
4-F	1/4 points	Machined Flush	6.04	2.00	0.490	0.141
4-G			12.02	2.00	0.489	0.141
4-H			18.02	2.00	0.489	0.141
4-J			24.03	2.00	0.489	0.141
4-K			30.01	2.00	0.489	0.141
4-L	ends and mid-length	Intact	6.00	2.00	0.490	0.141
4-M			12.05	2.00	0.489	0.141
4-N			18.19	2.00	0.489	0.141
4-O			24.03	2.00	0.490	0.141
4-P			30.04	2.00	0.489	0.141
4-Q	ends and mid-length	Machined Flush	6.01	2.00	0.489	0.141
4-R			12.05	2.00	0.489	0.141
4-S			18.16	2.00	0.489	0.141
4-T			24.04	2.00	0.490	0.141
4-U			30.05	2.00	0.489	0.141
5-A	mid-length	Intact	3.00	2.00	0.253	0.0731
5-B			6.00	2.00	0.254	0.0733
5-C			9.00	2.00	0.253	0.0731
5-D			12.00	2.00	0.254	0.0733
5-E			15.00	2.00	0.253	0.0731
5-F	mid-length	Machined Flush	3.00	2.00	0.210	0.0607
5-G			6.00	2.00	0.210	0.0607
5-H			9.00	2.00	0.208	0.0599
5-J			12.00	2.00	0.203	0.0585
5-K			14.99	2.00	0.207	0.0598
5-L	ends	Intact	3.11	2.00	0.254	0.0733
5-M			6.09	2.00	0.254	0.0733
5-N			8.90	2.00	0.254	0.0733
5-O			11.95	2.00	0.254	0.0733
5-P			15.22	2.00	0.254	0.0733
5-Q	ends	Machined Flush	3.13	2.00	0.254	0.0733
5-R			6.09	1.99	0.254	0.0733
5-S			8.92	2.00	0.253	0.0731
5-T			11.92	1.99	0.254	0.0733
5-U			15.22	1.99	0.254	0.0733
5-V	No Welds		3.00	2.00	0.253	0.0731
5-W			6.00	2.00	0.253	0.0731
5-X			9.00	2.00	0.254	0.0733
5-Y			12.00	2.00	0.253	0.0731
5-Z			15.00	2.00	0.254	0.0733

[a] 6061-T6 parent metal, 4043 filler metal.
[b] Could not be measured due to presence of spatter.
[c] K was assumed as 0.5 for fixed ends.

WELDED RECTANGULAR CROSS-SECTION COLUMNS

Extent of Reduced-Strength Zone, b_r in. (8)	Effective Slenderness Ratio[c] KL/r (9)	Failure Load, P, kips (10)	Column Strength f_c, ksi Measured (f_c = P/A) (11)	Column Strength f_c, ksi Computed (12)	Computed f_c / Measured f_c (13)	Maximum Crookedness in. (14)
0.62	21.9	32.40	32.7	31.8	0.97	0.0027
0.62	42.6	27.25	27.6	28.8	1.04	0.0041
0.62	63.9	21.80	22.2	22.2	1.00	0.0023
0.62	85.3	13.55	13.7	13.7	1.00	0.0036
0.62	106.4	8.55	8.7	8.7	1.00	0.0064
0.62	21.4	29.10	29.4	31.8	1.09	0.0030
0.62	42.6	25.65	26.0	28.8	1.11	0.0048
0.62	64.0	19.45	19.7	22.2	1.12	0.0054
0.62	85.3	13.10	13.3	13.7	1.03	0.0023
0.62	106.4	8.40	8.5	8.7	1.02	0.0047
0.62	21.3	25.45	25.8	20.5	0.79	0.0005
0.62	42.8	19.20	19.4	16.7	0.86	0.0014
0.62	64.5	15.10	15.3	14.1	0.93	0.0045
0.62	85.3	12.65	12.8	11.9	0.93	0.0017
0.62	106.6	8.85	9.0	8.7	0.97	0.0070
0.62	21.3	21.90	22.2	20.5	0.93	0.0019
0.62	42.8	16.20	16.4	16.7	1.02	0.0034
0.62	64.4	13.85	14.1	14.1	1.00	0.0037
0.62	85.3	11.10	11.2	11.9	1.06	0.0020
0.62	106.7	8.05	8.2	8.7	1.06	0.0060
0.53	20.5	16.94	33.4	36.2	1.09	—[b]
0.53	41.0	12.70	25.0	23.8	0.95	0.0037
0.53	61.6	9.38	18.5	17.3	0.94	0.0058
0.53	82.0	7.00	13.8	13.7	0.99	0.0049
0.53	102.6	4.71	9.3	9.5	1.02	0.0097
0.53	24.8	12.60	30.0	34.1	1.14	0.0016
0.53	49.4	8.68	20.6	20.0	0.97	0.0016
0.53	75.1	5.70	13.7	15.0	1.10	0.0061
0.53	102.6	3.83	9.5	9.5	1.00	0.0070
0.53	125.4	2.45	5.9	6.3	1.06	0.0015
0.53	21.2	17.32	34.1	36.0	1.05	0.0012
0.53	41.6	12.35	24.4	23.3	0.95	0.0060
0.53	60.7	9.60	18.9	17.5	0.93	0.0008
0.53	81.4	7.45	14.7	13.9	0.94	0.0028
0.53	103.9	4.70	9.3	9.3	1.00	0.0038
0.53	21.3	15.80	31.2	36.0	1.15	0.0001
0.53	41.6	12.00	23.8	23.3	0.98	0.0010
0.53	61.0	8.38	16.6	17.5	1.05	0.0017
0.53	81.3	7.00	13.8	13.9	1.01	0.0035
0.53	103.9	4.61	9.1	9.3	1.02	0.0037
0.53	20.5	20.72	40.9	39.6	0.97	0.0032
0.53	41.0	17.27	34.1	34.0	1.00	0.0022
0.53	61.5	12.45	24.6	25.5	1.04	0.0028
0.53	82.2	7.35	14.5	14.6	1.01	0.0065
0.53	102.4	4.83	9.5	9.5	1.00	0.0068

TABLE 3.—PROPERTIES OF TRANSVERSELY WELDED, TUBULAR CROSS SECTION COLUMNS[a]

Specimen No.	Location of Welds	Length, L, in.	Extent of Reduced-Strength Zone, b_r, in.	Fixity Coefficient, K	Effective Slenderness Ratio, KL/r	Failure Load, P kips	Column Strength, f_c, ksi		Computed / Measured f_c	Maximum Crookedness,[c] in.
							Measured $f_c = P/A$	Computed		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
6-A	none	41.5	0.00	1.00	42.5	79.2	36.7	37.1	1.01	0.004
6-B	none	41.5	0.00	1.00	42.5	79.2	36.7	37.1	1.01	0.002
6-C	center	41.5	1.03	1.00	42.5	40.2	18.6	18.5	0.99	0.004
6-D	center	41.5	1.03	1.00	42.5	43.6	20.2	18.5	0.92	0.003
6-E	ends	41.5	1.03	1.00	42.5	64.5	29.9	28.0[b]	0.94	0.003
6-F	ends	39.0	1.03	0.50	20.0	68.9	31.9	28.0[b]	0.88	0.004
6-G	center	11.5	1.03	0.50	5.9	71.0	32.8	28.0[b]	0.85	---

[a] 6061-T6, 5556 filler metal outside diameter = 3.0 in., wall thickness = 1/4", A = 2.160 in.^{2}, r = 0.976 in.
[b] Local buckling strength of welded regions.
[c] Maximum deviation of axis of column from straight line through ends.

TABLE 4.—AVERAGE MECHANICAL PROPERTIES OF WELDED COLUMN SPECIMENS

Alloy and Temper	Product	Filler Metal	Parent Metal Properties				Properties of Weakest Material in Reduced Strength Zone			
			Tensile Strength, ksi	Tensile Yield Strength, ksi	Elongation, per cent in 2 in.	Comp. Yield Strength, ksi	Tensile Strength, ksi	Tensile Yield Strength, ksi	Elongation, per cent	Comp. Yield Strength ksi
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
5456-H321	1/2 in. plate	5456	52.2	34.2	17.1	34.5	39.1	19.7	16.0	18.9
5154-H34	1/2 in. plate	5154	40.8	29.6	17.7	29.5	33.5	14.5	25.0[a]	15.4
6061-T6	1/2 in. plate	4043	45.9	43.0	19.6	42.1	34.3	17.8	10.0[a]	17.6
6061-T6	1/2 in. plate	4043	46.2	42.2	17.0	40.8	39.1	19.7	6.0[b]	19.0
6061-T6	3 in. O.D. by 1/4 in. Wall Tube	5556	47.1	41.2	17.0	43.4	--	--	--	14.2

[a] 1-in. gage length.
[b] 1/2 in. gage length.

column were adjusted until strain measurements indicated eccentricities of less than 0.005 in.

During the tests of the tubular columns, a continuous record of load versus bending strain was obtained as a check on the concentricity of the specimens. A similar check was made on the other columns by means of a continuous record of load versus lateral deflection. In obtaining these records, loads on the Amsler machine were measured with an SR-4 pressure cell in the load measuring system, while loads on the Baldwin machine were taken directly from the load sensing device with a rack and pinion. Bending strains were measured with SR-4 strain gages placed at the midlength of the specimen. The same gages were used for centering. Lateral deflections were measured with a small cantilever beam instrumented with SR-4 strain gages. The results of the deflection and strain measurements demonstrated that accidental crookedness and eccentricity for the specimens were of the same order of magnitude as the crookedness values measured before testing and listed in Tables 1, 2, and 3. These values are small enough to have a negligible effect on the column strength.

Hardness surveys were made on all the welded plates and tubes from which specimens were made. Tensile-property and residual-stress surveys were made on a few of the 36 in. by 48 in. plates from which the longitudinally welded specimens were made. The results of the hardness and tensile property surveys were used to determine values of b_r for use in the analyses. The satisfactory correlation between the computed curves and the test results demonstrates that values of b_r so determined are adequate for use in computing the strength of welded columns.

TEST RESULTS

Longitudinally Welded Columns.—The results of the tests of the longitudinally welded columns are reported in Table 1 and plotted in Figs. 3 through 5. Computed column strengths, also listed in Table 1, were all within 14% of measured values.

Residual-stress measurements for three of the welded plates, from which the longitudinally welded specimens were made, are discussed in Appendix I. The residual-tensile stresses in the vicinity of a weld can be of the order of magnitude of the yield strength of the heat-affected material. This is the case, however, only for the complete welded plates. By sawing the column specimens out of the larger welded plates, the residual stresses were considerably relieved, as is demonstrated by Fig. 14. The specimens for which the residual stresses are shown in these figures were, in each case, the widest specimens of the particular alloy and, thus, had the highest values of residual stress. Only in the case of specimens 3D and 3E of 6061-T6 were the residual stresses large enough to have an appreciable effect on the column strength.

For these specimens, two computed column curves are shown in Fig. 5 ($A_r/A = 0.29$). One curve includes the effect of residual stress, and the other does not. These curves demonstrate the effect of moderate residual stresses on a column having a relatively high proportion of the cross section affected by welding. The effects of residual stress were neglected in computing the other column curves of Fig. 3, 4, and 5.

The effect of larger values of residual stress on the strength of longitudinally welded columns was investigated analytically using minimum expected

properties for two alloy-temper combinations (6061-T6 and 5456-H321). The results of these computations are plotted in Figs. 12 and 13. The analysis was based on the assumption that the residual stress in the reduced-strength zone was a tensile stress equal to the yield strength of the most highly heat-affected material adjacent to the weld and that there was residual compressive stress over the rest of the cross section, sufficient to balance the residual tensile stresses.

The curves in Figs. 12 and 13 demonstrate that in most practical cases, residual stresses caused by welding have a beneficial effect on the strength of welded aluminum columns. This would not be the case for alloys that undergo a relatively small reduction in yield strength resulting from welding—annealed material, for example—or for columns that have a large percentage of the cross section within the reduced-strength zones. Figs. 12 and 13 indicate, however, that the effects of residual stresses can be safely neglected for 6061-T6 columns with $A_r/A \leq 0.5$ or for 5456-H321 columns with $A_r/A <$ about 0.3. The latter alloy is believed to be typical of the aluminum-magnesium alloys in this respect.

Transversely Welded Columns.—The results of the tests of the transversely welded columns are given in Tables 2 and 3 and plotted in Figs. 7, 8, and 9. Also plotted in these figures are the results of computations using stepped-column theory.

Three of the curves of Fig. 9 are shown dotted for stresses above 24 ksi because the compressive stress-strain measurements for the reduced-strength material of these specimens did not extend above 24 ksi. Beyond this stress, the strains became too high to measure with the SR-4 gages used. The stress-strain curves were extended beyond 24 ksi on the basis of data from tests of reduced-strength regions of other 6061-T6 weldments.

The failure stresses of the specimens are compared with the computed column strengths in Tables 2 and 3. With one exception, the computed strengths are within 15% of the measured values. In one case, the measured value was 21% higher than the computed value. This specimen, 4-L, was a 6-in. long fixed-end specimen containing two transverse welds (at the 1/4 points) and having the weld beads intact. Thus, the increase in cross section due to the presence of the weld beads, which was neglected in the computations, affected a large proportion of the length of the column. The presence of the weld beads apparently caused the discrepancy between measured and computed column strength, since the strength of the corresponding specimen with the weld beads ground flat was within 7% of the computed value.

With the exception of three of the tubular specimens, all specimens failed by general column buckling. Three of the tubular specimens failed by local buckling. The results of these three tests are indicated by the solid points in Fig. 9. Two of these specimens contained welds at the ends. One of these was tested in the pinned-end condition and one as a fixed-end column. The third specimen was a short fixed-end specimen containing a single weld at midlength. The dotted horizontal line in Fig. 9 indicates the computed local buckling stress for the reduced-strength material of the tubular columns. This was obtained by computing an effective slenderness ratio for local buckling in accordance with the Alcoa Structural Handbook,[17] and finding the stress corresponding to this slenderness ratio on the column curve for an all reduced-strength column. The fact that the actual local buckling strengths exceeded this computed value

[17] Alcoa Structural Handbook, Anon., Aluminum Company of America, Pittsburgh, Pa., 1958, Eq. 24, p. 156.

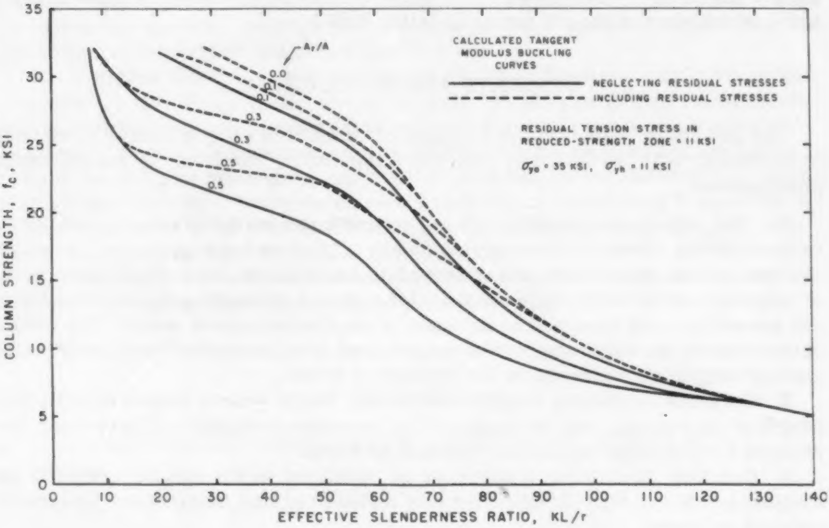


FIG. 12.—EFFECT OF RESIDUAL STRESSES ON STRENGTH OF 6061-T6 WITH MINIMUM MECHANICAL PROPERTIES

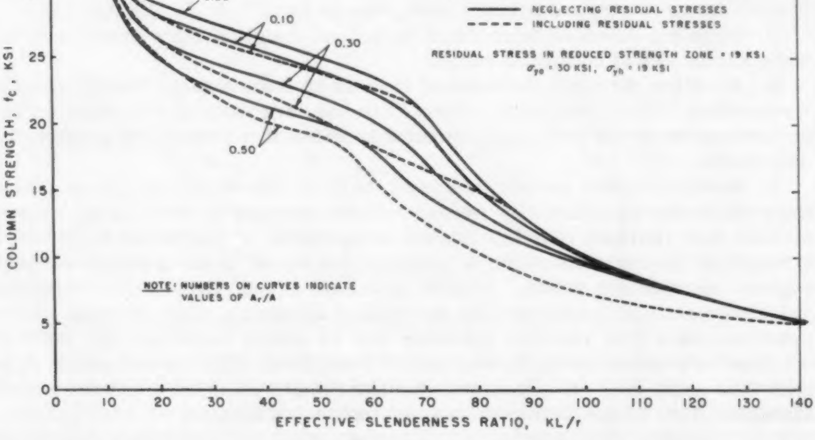


FIG. 13.—EFFECT OF RESIDUAL STRESSES ON STRENGTH OF LONGITUDINALLY WELDED COLUMNS OF 5456-H321 WITH MINIMUM MECHANICAL PROPERTIES

was probably due in part to the restraint offered by the adjacent unaffected parent material. Additional restraint was offered by the weld beads, which were of a higher-strength material (alloy 5556).

SUMMARY AND CONCLUSIONS

The effects of welds on the strength of aluminum alloy columns have been investigated both analytically and experimentally, resulting in the following conclusions:

1. The effects of a weld on column strength can be satisfactorily evaluated by considering the weld to be surrounded by a "reduced-strength zone" in which the mechanical properties are assumed to be those of the softest material in or adjacent to the weld, while outside this reduced-strength zone, the mechanical properties are assumed to be those of unaffected parent metal. The extent of the reduced-strength zone can be determined from surveys of tensile strength, yield strength, or hardness in the vicinity of welds.
2. Columns containing longitudinal welds, which extend essentially the full length of the column, can be analyzed by the same methods that have been developed for columns containing residual stresses.
3. Columns containing transverse or localized welds can be analyzed as stepped columns, with the transversely welded regions being short lengths of reduced stiffness.
4. Formulas for buckling stress for transversely welded columns can be put in a form that equates the slenderness ratio of the column to the slenderness ratio of a homogeneous parent metal column that would buckle at the same stress, minus a correction factor that takes account of the effect of the transverse welds. The correction factor for any number of arbitrarily located transverse welds can be considered as the sum of the correction factors for the individual welds, by the use of Eqs. 9, 10, and 11. Solutions for various special cases of stepped columns are given by Eqs. 7, 8, 12, 13, and 14.
5. Buckling stresses determined in tests of longitudinally welded columns were within 14% of computed values.
6. Buckling stresses determined in tests of transversely welded columns were within 15% of computed values, with the exception of one short column containing two welds with beads intact that failed at a stress 21% greater than calculated.
7. Residual stress measurements on the 36 in.-by-48 in.-by-1/2-in. plates, from which the longitudinally welded column specimens were made, demonstrated that residual tensile stresses comparable in magnitude to the yield strength of the reduced-strength material can occur in the reduced-strength regions surrounding welds. In most practical cases, the effect of these residual stresses is to increase the strength of aluminum alloy columns. Computations show that residual stresses can be safely neglected for 6061-T6 columns with values of A_r/A less than 0.5 and 5456-H321 columns with A_r/A less than about 0.3. It is believed that the results for 5456-H321 are representative of all of the aluminum-magnesium series alloys.

On the basis of this investigation, the following recommendations are made for design of welded aluminum alloy columns.

A. Longitudinally welded columns can be designed on the basis of families of column curves, with the individual curves of the family corresponding to various values of A_r/A. The column curves are computed from the tangent

modulus formula on the basis of weighted average compressive stress-strain curves. For a given strain, the stress on the weighted average stress-strain curves is the weighted average of the stress in the reduced-strength zone and the stress in the unaffected parent metal.

B. Design of transversely welded columns can be based on a stepped-column analysis. To use the stepped-column formulas directly in design would require that an "allowable" slenderness ratio be computed for a given column and compared with the actual slenderness ratio. Another approach is to use the stepped-column analysis in developing approximate procedures for determining allowable stress directly. Transverse welds whose reduced-strength zones do not extend more than 0.05 L from the ends of the column have a negligible effect on the buckling strength of pinned-end columns.

C. The effect of residual stresses on the strength of welded aluminum alloy columns can be neglected in most practical cases.

APPENDIX I.—RESIDUAL STRESSES IN WELDED PLATES

The residual stresses in three of the welded plates from which the longitudinally welded column specimens were sawed are plotted in Fig. 14. The residual stresses were determined by converting relaxation strains measured

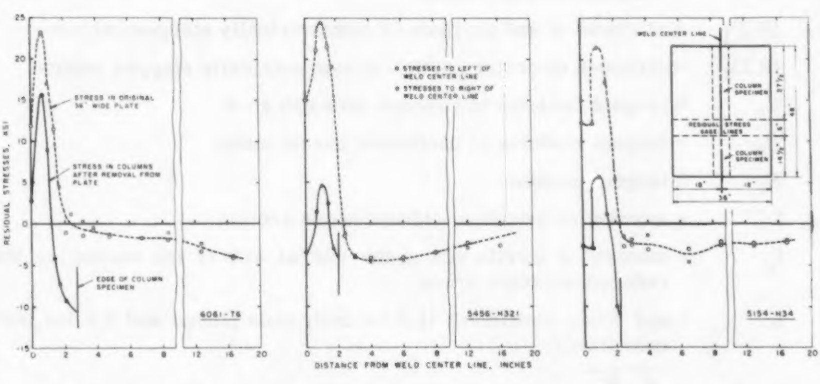


FIG. 14.—RESIDUAL STRESSES DUE TO WELDING IN 1/2 INCH PLATE

with a 2-in. gage length Berry strain gage. Gage lines were located parallel to the weld at various distances from the weld. These gage lines were then isolated from the large plate by sawing, and the relaxation strains measured. The residual stresses were determined by multiplying these relaxation strains by Young's modulus for the material, negative strains indicating residual tension and positive strains indicating residual compression. This procedure involves the assumption that residual stresses transverse to the weld are negligible. Other investigations have demonstrated this to be essentially true for all locations except at the ends of the welds.

The residual stresses in the narrow strip containing the weld, from which the specimens were machined, were also determined. These are plotted as the solid lines in Fig. 14.

Comparison of the solid lines with the corresponding dotted lines demonstrates the extent to which the as-welded residual stresses were reduced in the process of sawing out the specimens. The residual stresses were, no doubt, further reduced by machining 1/4 in. off each side of the as-sawed column specimens. The specimens for which Fig. 14 are drawn are the widest column specimens for each alloy. For the other specimens, the residual tensile stresses would be even smaller and the compressive stresses only slightly larger.

APPENDIX II.—NOTATION

A = area of column cross section

A_r = reduced-strength area of the cross section

b_r = extent of reduced-strength zone

$(E\,I)_h$ = stiffness of sections of transversely welded column within reduced-strength zones

$(E\,I)_o$ = stiffness of sections of transversely welded column outside reduced-strength zones

$(E\,I)_1$ = stiffness of end portions of symmetrically stepped column

$(E\,I)_2$ = stiffness of center portion of symmetrically stepped column

E_h = tangent modulus in reduced-strength zone

E_o = tangent modulus of unaffected parent metal

E_t = tangent modulus

I = moment of inertia of column cross section

I_r = moment of inertia (about the neutral axis of the section) of the reduced-strength areas

K = end fixity coefficient (1.0 for both ends pinned and 0.5 for both ends fixed)

$k_1 = \sqrt{\dfrac{P}{(E\,I)_1}}$

$k_2 = \sqrt{\dfrac{P}{(E\,I)_2}}$

L = length of column

L_h = length of prismatic column of reduced-strength material, with buckling load P

L_o = length of prismatic unaffected parent metal column with buckling load P

L_r = reduced-strength length for a given weld, (length of column contained in the reduced-strength zone for a given weld)

L_1 = length of an end portion of symmetrically stepped column

L_2 = half-length of a stiffened portion of symmetrically stepped column

P = buckling load of column

r = radius of gyration of column cross section

ϵ = strain

σ_h = stress in reduced-strength zone

σ_o = stress outside reduced-strength zone

σ_c = column buckling stress

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SOME BASIC CONCEPTS IN MATRIX STRUCTURAL ANALYSIS

By Frank R. Berman[1]

SYNOPSIS

The development of the electronic digital computer and the application of matrix algebra has made it possible for the structural engineer to analyze complex or highly redundant structures which previously had been prohibitive to analyze by the hand method of calculation. Papers recently written have utilized either the "flexibility" or "stiffness" matrix approach. It is the purpose of this paper to point out the common origin of these methods in energy and to discuss procedures for calculating the total energy of a structure. The concept of the "total" structure will be considered. Applications include analysis of a truss with elastic supports and a continuous beam.

INTRODUCTION

The analysis of complex or highly redundant structures which had been prohibitive to analyze by hand calculation became a possibility as a result of the development of the electronic computer. Furthermore, the application of matrix theory to structural analysis was then logical and permitted the concise formulation of large problems and control of the data required for submission to the electronic computer.

Many writers (2,4-20)[2] have shown the matrix formulation of structural theory and its application to structural problems.

Note.—Discussion open until January 1, 1961. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Structural Engineering Division, Proceedings of the American Society of Civil Engineers, Vol. 86, No. ST 8, August, 1960.

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[2] Numerals in parentheses, thus (1), refer to corresponding items in the Bibliography.

In this country, S. Levy's (3) application in 1947 of the strain energy theory to a complex aircraft structure was presented in matrix form by R. L. Bisplinghoff and A. L. Lang (8). W. Lansing and L. B. Wehle (10), P. H. Denke (13), R. W. Clough (17), B. Klein (18), H. C. Martin (19), Turner (21), and many others (2,16) have likewise made noteworthy contributions.

In Europe, B. Langefors (7, 9, 12), starting with the tensor concepts of G. Kron (1), independently pioneered in the matrix formulation of structural theory. Papers by H. Falkenheimer (5) and others (6, 15) followed. Probably the most comprehensive and thorough treatment of energy methods in matrix and non-matrix form is that given by J. H. Argyris (15).

The concept of energy is the starting point of the paper, but first we will note a superposition relationship that we will need for our work.

SUPERPOSITION OF DEFLECTIONS

Given a structure in the plane acted upon by a set of forces P_1, P_2, P_3 in equilibrium with reaction forces R_1, R_2, and R_3 (Fig. 1), we assume that linearity and the law of superposition apply. Thus, if Δ_1 is the deflection in the direction of P_1 we can write

$$\Delta_1 = \delta_{11} P_1 + \delta_{12} P_2 + \delta_{13} P_3 \quad \text{. . . . .} \quad (1)$$

We can generalize Eq. 1 for a set of forces P_1, P_2, . . . P_n with the result

$$\Delta_i = \delta_{i1} P_1 + \delta_{i2} P_2 + \ldots \delta_{ii} P_i + \ldots \delta_{in} P_n \quad \text{. . . . . .} \quad (2)$$

or in matrix form

$$\left\{ \Delta_i \right\} = \left[\delta_{ij} \right] \left\{ P_j \right\} \quad \text{. . . . .} \quad (3)$$

We recognize in Eq. 3 that $\left[\delta_{ij} \right]$ is the matrix of influence deflection coefficients which in the literature is also known as a "flexibility" matrix.

We can now examine the basic concept of strain energy.

STRAIN-ENERGY CONCEPT

Consider the axially compressed bar, shown in Fig. 2. If we define the strain energy as the work done by the applied force, P_1, in shortening the bar by the amount, Δ_1, we can write, noting that P_1 builds up from zero to its final value of P_1,

$$U = \frac{1}{2} P_1 \Delta_1 \quad \text{. . . . .} \quad (4)$$

Now let us generalize Eq. 4. Consider the linear elastic planar structure given in Fig. 3. The loads P_1, P_2, . . . P_n, shown acting, are in equilibrium and include both the applied loads and their reactions. If we measure all deflections with respect to an arbitrary set of orthogonal axes (shown as x, y axes), we have the deformed shape of the structure as shown in Fig. 3.

If we define the strain energy as the work done by all the applied forces P_1, P_2, . . . P_n in deforming the structure and noting that all the forces increase

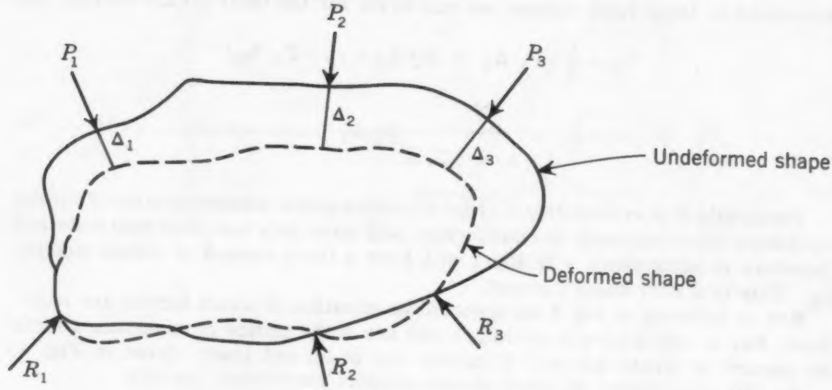

FIG. 1

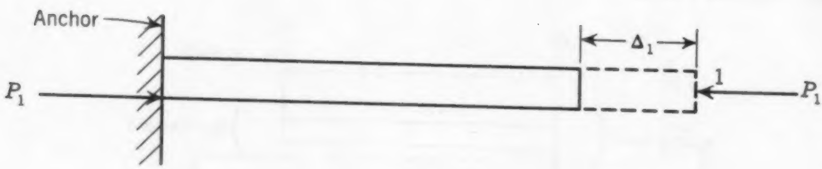

FIG. 2

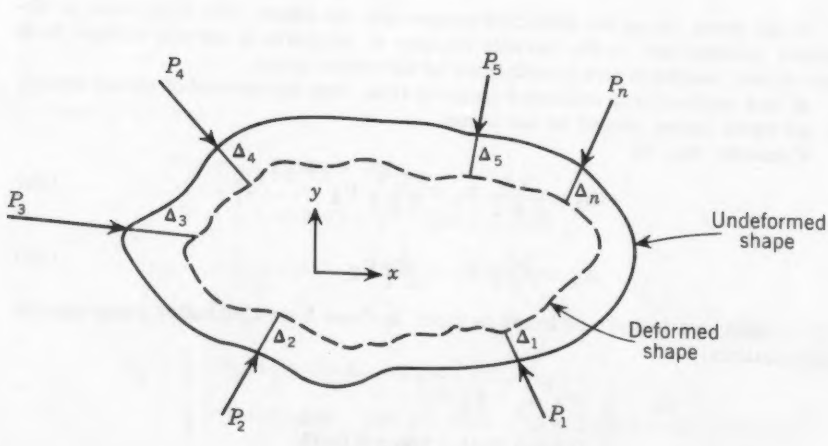

FIG. 3

from zero to their final values, we can write for the total strain energy, U_S,

$$U_S = \frac{1}{2}\left(P_1 \Delta_1 + P_2 \Delta_2 + \ldots P_n \Delta_n\right)$$

$$= \frac{1}{2} \sum_{i = 1, 2, \ldots n} P_i \Delta_i \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(5)}$$

Physically it is evident that a given structure under a known system of forces (including the reactions), in equilibrium, will have only one deformed state and therefore in accordance with Eq. 5 will have a fixed amount of strain energy, U_S. This is a very basic concept.

Now in arriving at Eq. 5 we ignored the question of which forces are reactions. Let us use a simple structure and see if the choice of reactions affects the amount of strain energy. Consider the beam and loads shown in Fig. 4. We will study three different simple support conditions; namely,

(1) Case I - Cross section at (a) is clamped.
(2) Case II - Cross section at (c) is clamped.
(3) Case III - Cross section at (b) is clamped.

Consider first the deflected shapes. Fig. 5 shows the probable shapes for the different cases.

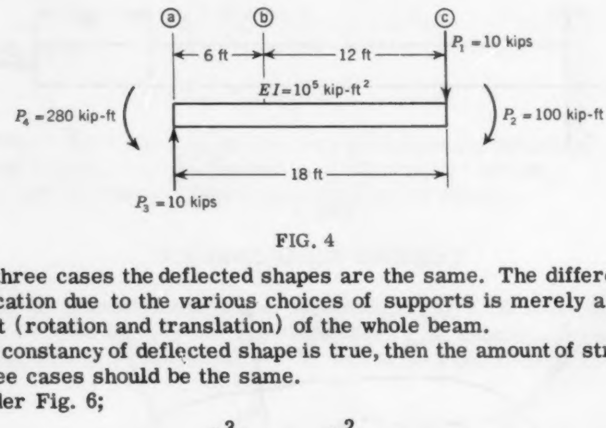

FIG. 4

In all three cases the deflected shapes are the same. The difference in absolute location due to the various choices of supports is merely a rigid body movement (rotation and translation) of the whole beam.

If this constancy of deflected shape is true, then the amount of strain energy in all three cases should be the same.

Consider Fig. 6;

$$\Delta_1 = \frac{L^3}{3\,E\,I} P_1 + \frac{L^2}{2\,E\,I} P_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(6a)}$$

$$\Delta_2 = \frac{L^2}{2\,E\,I} P_1 + \frac{L}{E\,I} P_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(6b)}$$

All lengths are in feet and loads in kips. In Case I, we find (after some simple calculations),

$$\begin{aligned} U_S &= \frac{1}{2}\left[P_1 \Delta_1 + P_2 \Delta_2\right] \\ &= \frac{1}{2}\left[10 \times 0.3564 + 100 \times 0.0342\right] \\ &= \frac{1}{2}\left[3.564 + 3.42\right] = 3.492 \text{ kip-ft} \end{aligned}$$

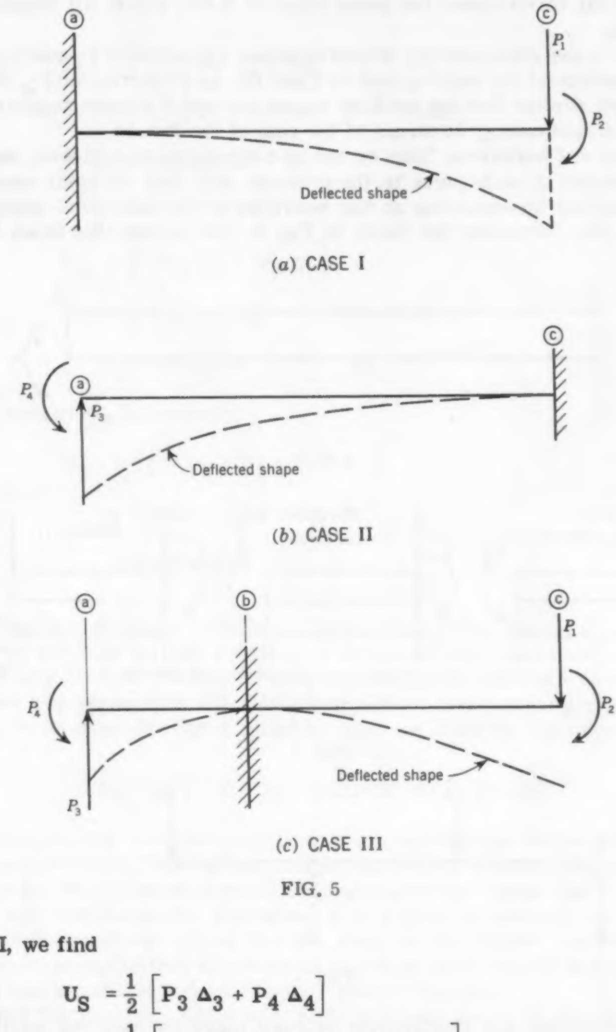

FIG. 5

In Case II, we find

$$U_S = \frac{1}{2}\left[P_3\,\Delta_3 + P_4\,\Delta_4\right]$$
$$= \frac{1}{2}\left[-0.2592 \times 10 + 280 \times 0.0342\right]$$
$$= \frac{1}{2}\left[-2.592 + 9.576\right] = 3.492 \text{ kip-ft}$$

In Case III, we find

$$U_S = \frac{1}{2}\left[P_1\,\Delta_1 + P_2\,\Delta_2 + P_3\,\Delta_3 + P_4\,\Delta_4\right]$$
$$= \frac{1}{2}\left[10 \times 0.1296 + 100 \times 0.0192 - 10 \times 0.0432 + 280 \times 0.0150\right]$$
$$= \frac{1}{2}\left[1.296 + 1.92 - 0.432 + 4.20\right] = 3.492 \text{ kip-ft}$$

We see in all three cases the same value of 3.492 kip-ft for the total strain energy, U_S.

In Case I, the strain energy is expressed as a function of P_1 and P_2; in Case II, as a function of P_3 and P_4; and in Case III, as a function of P_1, P_2, P_3 and P_4. We now realize that we are free to use any set of simple supports and express the strain energy in terms of the rest of the forces.

Next, we ask ourselves "can we cut up a structure into pieces, maintaining the equilibrium of each piece in the process, and find the total energy of the uncut structure by summing up the energies of the individual pieces?" The answer is yes. Consider the beam in Fig. 4. Let us cut this beam into three

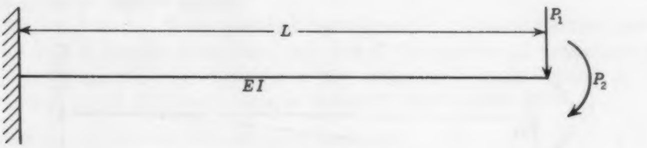

FIG. 6

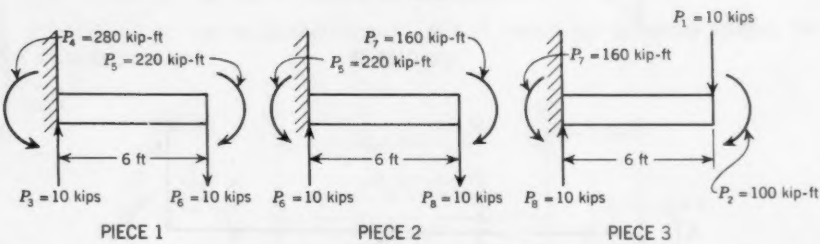

FIG. 7

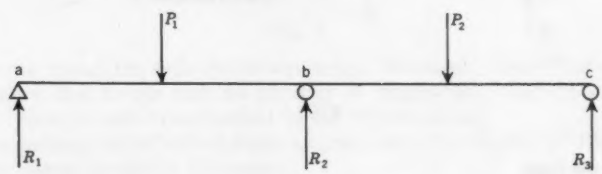

FIG. 8

pieces, maintaining the equilibrium of each piece by applying equilibrating forces, as shown in Fig. 7.

Each piece will have the same deformed shape that it had as part of the uncut structure. Accordingly, the sum of the energies of the individual pieces should equal the total energy of the uncut structure.

Each piece, for illustration, is clamped along its left edge. Using formulas similar to Eqs. 6, we find for piece 1:

$$\begin{aligned} U_1 &= \frac{1}{2}\left[P_6\,\Delta_6 + P_5\,\Delta_5\right] \\ &= \frac{1}{2}\left[+\,10 \times 0.0468 + 220 \times 0.0150\right] \\ &= 1.884 \text{ kip-ft} \end{aligned}$$

For piece 2:

$$U_2 = \frac{1}{2}\left[P_7\,\Delta_7 + P_8\,\Delta_8\right]$$
$$= \frac{1}{2}\left[+ 160 \times 0.0114 + 10 \times 0.0360\right]$$
$$= 1.092 \text{ kip-ft}$$

For piece 3:

$$U_3 = \frac{1}{2}\left[P_1\,\Delta_1 + P_2\,\Delta_2\right]$$
$$= \frac{1}{2}\left[10 \times 0.025 + 100 \times 0.0078\right]$$
$$= 0.516 \text{ kip-ft}$$

The total energy U_S is given by

$$U_S = U_1 + U_2 + U_3$$
$$= 1.884 + 1.092 + 0.516$$
$$= 3.492 \text{ kip-ft}$$

which checks our previous calculation of the total strain energy.

Thus, we see that we can break up a structure into convenient units. We can select any group of reactions or simple supports for each piece. Before we proceed with the matrix formulation of theory incorporating the preceding concepts, let us consider for a moment what we mean by the total structure.

CONCEPT OF THE "WHOLE" STRUCTURE

Traditionally, the structural engineer has concentrated his analysis efforts on the superstructure. Noting that interaction with the foundation does influence the stress and displacements within the superstructure in the cases of yielding supports and indeterminate reactions, it is logical to include the foundation with the superstructure calling this combination the "whole" structure. This is tantamount to saying that all elements that may contribute to the total energy of the system should be included in the "whole" structure.

Consider the two-span beam on elastic supports shown in Fig. 8.

To show the inclusion of the foundation, we redraw Fig. 8 with the results shown in Fig. 9.

The foundation member, shown in Fig. 9, has the same elastic effects (that is, elastic support movements) at the beam supports as specified in the original problem. If in Fig. 8 the support at a settled an amount S due to a given value of R_1, then the foundation member shown in Fig. 9, would displace the same amount S at a due to the same value R_1, acting upon the foundation member at a. The simple reactions shown at point o are those necessary for equilibrium of the external forces, such as P_1 and P_2. The addition of the foundation member now places the support forces R_1, R_2 and R_3 in the category of internal forces. Point o, selected as the point of application of the "whole"

structure reactions, is the arbitrary origin of a set of axes that neither translates nor rotates (and therefore the reactions do not contribute to the energy).

In Fig. 10, we have separated the superstructure from the foundation to facilitate the calculation of the total energy, U_T.

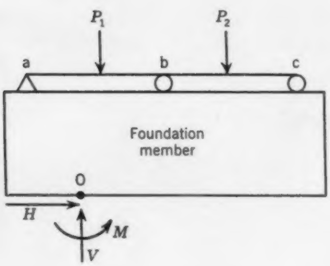

FIG. 9

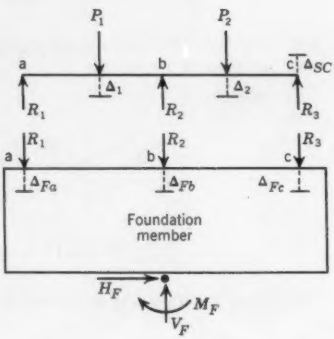

FIG. 10

We can write the energy U_S for the superstructure as

$$U_S = \frac{1}{2}\left[P_1\,\Delta_1 + P_2\,\Delta_2 + R_3\,\Delta_{SC}\right] \qquad (7)$$

and the energy of the foundation member, U_F, as

$$U_F = \frac{1}{2}\left(R_1\,\Delta_{Fa} + R_2\,\Delta_{Fb} + R_3\,\Delta_{Fc}\right) \qquad (8)$$

The total energy of the "whole" structure, U_T, is then given by

$$U_T = U_S + U_F \qquad (9)$$

Now let us proceed with the matrix formulation of theory beginning with energy expressions.

MATRIX FORMULATION OF ENERGY AS A FUNCTION OF LOADS

The matrix formulation of strain energy as a function of loads resulting in Eq. 21 (to be presented subsequently) has been covered elsewhere (8-10) and is repeated here both for completeness and to illustrate the fundamental concepts given in this paper.

We consider a "whole" structure acted upon by a set of forces (including reactions), P_S. Now we break the structure up into k convenient parts, maintaining the equilibrium of each part in the process by the addition of appropriate equilibrating forces at the cuts.

If each piece is considered to be in equilibrium under its own set of forces (including its reactions), P_{ki}, then we can express the energy of a typical piece, U_k, as

$$U_k = \frac{1}{2} \sum_{i = 1, 2, \ldots i} P_{ki}\,\Delta_{ki} \qquad (10)$$

where Δ_{ki} is the deflection of the P_{ki} load. The energy of the whole system, U_T, is the sum of the individual energies and is given by

$$U_T = U_1 + U_2 + \ldots U_k = \sum_{k = 1,2,k} U_k \quad \ldots\ldots\ldots\ldots \quad (11)$$

In matrix form we can rewrite Eq. 10 as

$$\left[U_k\right] = \frac{1}{2} \left[P_{ki}\right]\left\{\Delta_{ki}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

If we use Eq. 3 in Eq. 12, employing appropriate notation, we have as a result

$$\left[U_k\right] = \frac{1}{2} \left[P_{ki}\right] \left[\delta_{kij}\right] \left\{P_{kj}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

Eq. 13 expresses the strain energy of piece k as a function of the forces, P_{ki}, acting upon it. In Eq. 13, $\left[P_{ki}\right]$ is a row matrix the the transpose of $\left\{P_{kj}\right\}$. The term $\left[\delta_{kij}\right]$ is the matrix of flexibilities for piece k.

Now let us take a statically indeterminate structure under the action of applied loads, P_A. If we make the structure statically determinate by the selection of certain redundants, P_x, and consider the "whole" structure, that is, include the foundation, then we can write the total energy as

$$U_T = \frac{1}{2} \left[P_x \mid P_A\right] \left[\begin{array}{c|c} \delta_{xy} & \delta_{xB} \\ \hline \delta_{Ay} & \delta_{AB} \end{array}\right] \left\{\begin{array}{c} P_y \\ \hline P_B \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (14)$$

The use of partitioning here is to separate the redundant forces from the given applied loads acting upon the original structure. Eq. 14 is the desired equation for the total energy. Let us see how we may obtain the total energy by breaking up the statically determinate structure into k parts (including the foundation), and then summing up the energies of the individual parts as indicated by Eq. 11.

Starting with a typical k piece, we note that the loads, P_{ki}, acting upon the piece can be expressed in terms of the redundants, P_x, and the applied loads, P_A, acting upon the given structure; thus,

$$\left\{P_{ki}\right\} = \left[a_{kix} \mid a_{kiA}\right] \left\{\begin{array}{c} P_x \\ \hline P_A \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (15)$$

where the partitioned [a] represents the required coefficients expressing the P_{ki} in terms of the P_x and P_A loads. If we now use Eq. 15 in Eq. 13 we get for each k piece

$$\left[U_k\right] = \frac{1}{2} \left[P_x \mid P_A\right]\left[a_{kix} \mid a_{kiA}\right]^T \left[\delta_{kij}\right] \left[a_{kjy} \mid a_{kjB}\right] \left\{\begin{array}{c} P_y \\ \hline P_B \end{array}\right\} \quad \ldots\ldots \quad (16)$$

where $[\ ]^T$ stands for the transpose matrix. If we now use Eq. 16 in Eq. 11 we have as a result

$$\left[U_T\right] = \frac{1}{2} \left[P_x \mid P_A\right] \sum_k \left(\left[a_{kix} \mid a_{kiA}\right]^T \left[\delta_{kij}\right] \left[a_{kjy} \mid a_{kjB}\right]\right) \left\{\begin{array}{c} P_y \\ \hline P_B \end{array}\right\} \ldots \quad (17)$$

Let us now examine the expression

$$\left[a_{kix} \,\vdots\, a_{kiA}\right]^T \left[\delta_{kij}\right] \left[a_{kjy} \,\vdots\, a_{kjB}\right] \text{ .... } (18)$$

which appears in Eqs. 16 and 17. The term $\left[\delta_{kij}\right]$ is the matrix of influence deflection or flexibility coefficients for piece k in terms of the P_{ki} forces acting upon that piece. To transform these coefficients into coefficients in terms of the P_x and P_A forces acting upon the statically determinate structure we carry out the classical transformation given in Expression 18 which is of the form

$$[A]^T [B] [A] = [C] \text{ .... } 19$$

where $[B]$ is a symmetrical matrix; $[C]$, the result will be symmetrical, automatically.

Carrying out the triple matrix multiplication called for in Expression 18 results in

$$\left[a_{kix} \,\vdots\, a_{kiA}\right]^T \left[\delta_{kij}\right] \left[a_{kiy} \,\vdots\, a_{kjB}\right] = \left[\begin{array}{c|c} \delta_{kxy} & \delta_{kxB} \\ \hline \delta_{kAy} & \delta_{kAB} \end{array}\right] \text{ .... } (20)$$

which is the matrix of flexibilities for piece k in terms of the P_x and P_A forces. Using Eq. 20 in Eq. 17 we see that the total energy is given by

$$\left[U_T\right] = \frac{1}{2}\left[P_x \,\vdots\, P_A\right] \left(\sum_k \left[\begin{array}{c|c} \delta_{kxy} & \delta_{kxB} \\ \hline \delta_{kAy} & \delta_{kAB} \end{array}\right]\right) \left\{\begin{array}{c} P_y \\ \hline P_B \end{array}\right\} \text{ .... } (21)$$

Eq. 21 indicates a basic procedure for the determination of the total strain energy as the summation of the flexibilities of the individual parts expressed in terms of the redundants, P_x, and the given applied forces, P_A. The theory given by Eq. 20 would be used to transform the flexibilities given in terms of forces acting upon the k part into flexibilities in terms of P_x and P_A.

For an alternate procedure for evaluating the total energy, U_T, consider the summation of the triple products given in Eq. 17. We can write the triple product as

$$\sum_k \left[a_{kix} \,\vdots\, a_{kiA}\right]^T \left[\delta_{kij}\right] \left[a_{kjy} \,\vdots\, a_{kjB}\right] = [a]^T \left[\delta_k\right] [a] \text{ ..... } (22)$$

where $\left[\delta_k\right]$ is the diagonal block matrix

$$\left[\delta_k\right] = \begin{bmatrix} \delta_{1ij} & & & & & \\ & \delta_{2ij} & & & & \\ & & \delta_{3ij} & & & \\ & & & \delta_{4ij} & & \\ & & & & \ddots & \\ & & & & & \delta_{kij} \end{bmatrix} \text{ .... } (23)$$

and $[a]^T$ is the transpose of $[a]$ where $[a]$ is given by

$$[a] = \begin{bmatrix} a_{1jy} & a_{1jB} \\ a_{2jy} & a_{2jB} \\ a_{3jy} & a_{3jB} \\ a_{4jy} & a_{4jB} \\ \vdots & \vdots \\ a_{kjy} & a_{kjB} \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (24)$$

Matrix $[a]$ is the array of coefficients expressing the forces acting upon the pieces (such as bar forces) in terms of the redundants P_x and applied forces P_A. Having set up the $[\delta_k]$ and $[a]$ matrices, we use Eq. 22 with the result

$$[a]^T [\delta_k][a] = \left[\begin{array}{c|c} \delta_{xy} & \delta_{xB} \\ \hline \delta_{Ay} & \delta_{AB} \end{array}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (25)$$

Eq. 25 is used in Eq. 14 which is the expression for the total energy, U_T. It is to be noted that the size of matrices developed in building up the $[\delta_k]$ and $[a]$ matrices, in accordance with the foregoing theory in practice, is minimized by various packing or condensing techniques.

DEFLECTIONS AND THE METHOD OF LEAST WORK

If the total energy of a linear elastic structure is expressed as a function of one or more P_i loads, then using the well-known differential relation of Castigliano

$$\frac{\partial U_T}{\partial P_i} = \Delta_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (26)$$

we can find the deflection at P_i in the direction of P_i. Furthermore, if the P_i loads are taken as the redundants P_x, then $\Delta_x = 0$ in the usual case (keeping in mind, we are using the "whole" structure concept) and Eq. 26 becomes

$$\frac{\partial U_T}{\partial U_x} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (27)$$

which is known as the Least Work Theorem of Castigliano.

Applying Eq. 27 to a statically indeterminate structure whose total energy is given by Eq. 14 we find

$$\left\{\frac{\partial U_T}{\partial P_x}\right\} = \{0\} = \left[\delta_{xy} \;\vdots\; \delta_{xB}\right] \left\{\begin{array}{c} P_y \\ \hline P_B \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (28)$$

Eq. 28 is the matrix form of the equations of consistent deflection, known as the Maxwell-Mohr Equations. Solution of the x equations given by Eq. 28 yields for

the redundants

$$\{P_y\} = -[\delta_{xy}]^{-1}[\delta_{xB}]\{P_B\} = [D]\{P_B\} \qquad (29)$$

Eq. 29 expresses the redundant P_x (or P_y) in terms of the applied forces P_A (or P_B).

Stresses.—If we set up a matrix $[J]$

$$[J] = \left[\frac{D}{I_A}\right] \qquad (30)$$

then to find the forces acting upon the pieces of structure, we have

$$\{P_i\} = [a]\ [J]\ \{P_B\} \qquad (31)$$

To find any desired stresses within the original structure, we first find these stresses in terms of the redundants, P_x, and applied loads, P_A, thus

$$\{S\} = [b_x \mid b_A] \left\{\frac{P_x}{P_A}\right\} \qquad (32)$$

where $[b]$ is the matrix of coefficients expressing the stresses, S, in terms of the redundants and loads. Use of Eq. 30 in Eq. 32 gives the desired result for, S,

$$\{S\} = [b_x \mid b_A]\ [J]\ \{P_A\} \qquad (33)$$

Deflections.—To find deflections at the applied loads, P_A, we apply Eq. 26 to the expression for the total energy given by Eq. 14, to result in

$$\left\{\frac{\partial U_T}{\partial P_A}\right\} = \{\Delta_A\} = [\delta_{Ay} \mid \delta_{AB}] \left\{\frac{P_y}{P_B}\right\} \qquad (34)$$

In Eq. 34 the deflections $\{\Delta_A\}$ are expressed in terms of the redundants and applied loads. To find the deflections as a function of the applied loads only, we use Eq. 30 in Eq. 34 getting

$$\{\Delta_A\} = [\delta_{Ay} \mid \delta_{AB}]\ [J]\ \{P_B\} \qquad (35)$$

Eq. 35 is the desired result.

Flexibilities.—It should be noted that the flexibility coefficients for the original structure (in terms of the applied loads only) is given by

$$[\delta_{AB}] = [\delta_{Ay} \mid \delta_{AB}]\ [J] \qquad (36)$$

We see from the foregoing that Eq. 14 for the total energy, U_T, of the "whole" structure is the fundamental equation and the matrix contained in (21); namely,

$$\left[\begin{array}{c|c} \delta_{xy} & \delta_{xB} \\ \hline \delta_{Ay} & \delta_{AB} \end{array}\right]$$

is the key in the solution of the stresses and displacements in a structural problem.

To illustrate the theory and basic concepts, we will carry out two problems, the first involving a statically determinate truss on elastic supports and the second involving a continuous beam on unyielding supports.

PROBLEM 1

Given the statically determinate truss on elastic supports, shown in Fig. 11, it is desired to obtain the influence deflection coefficients for the truss, including the elasticity of the foundation. Basic data, including bar stresses for one kip values of the P loads, are given in Table 1. Values of the reactions R_1, R_2 and R_3 for one kip values of the loads are given in Table 2.

The flexibility coefficients, representing the elasticity of the foundation, are given in Table 3. It is to be noted that since the modulus of elasticity, E, is assumed constant at 30,000 ksi, it does not appear in any of these coefficients (or, in other words, all deflections are multiplied by E). We will drop the E in all subsequent formulation, and then in the final result reinsert it. For example, the deflection at R_2, Δ_{R_2} (in feet), is given as

$$E\,\Delta_{R_2} = 21.8081\,R_2 + 16.2851\,R_3 \qquad (37)$$

Let us now draw the "whole" structure (Fig. 12).

We will designate the energy of the superstructure by U_S and that of the foundation by U_F. The basic procedure to obtain the total energy U_T (and therefore our answer) follows three steps (noting that no redundants exist in this problem):

(1) The truss is considered to be broken up into the nine basic bars making up the truss. These are the k pieces cited previously. The energy, U_S, is obtained using the alternate procedure which essentially consists of setting up one matrix incorporating all the flexibilities of the 9 bars as indicated by Eq. 23 and then finding the energy, U_S.

(2) The energy, U_F, of the foundation is found from Eq. 16 using the matrix flexibility of the foundation which essentially is Table 2.

(3) The total energy, U_T, is found using the procedure indicated in Eqs. 11 and 21 and is given by

$$U_T = U_S + U_F \qquad (38)$$

The answer to our problem is the flexibility matrix, used to define the total energy, U_T. This matrix will yield deflections relative to absolute reference axes located at point o in Fig. 12.

Step 1.—Using the data in Table 1, the flexibility matrix $\left[\delta_k\right]$ given by Eq. 23 is

$$\left[\delta_k\right] = \begin{bmatrix} 1.748 &&&&&&&& \\ & 1.748 &&&&&&& \\ && 1.748 &&&&&& \\ &&& 1.748 &&&&& \\ &&&& 1.512 &&&& \\ &&&&& 1.512 &&& \\ &&&&&& 1.512 && \\ &&&&&&& 2.867 & \\ &&&&&&&& 2.867 \end{bmatrix} \qquad (39)$$

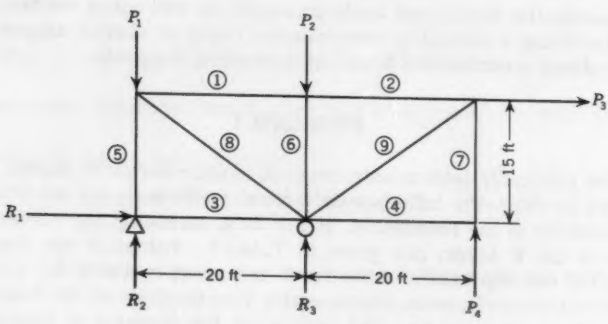
P_1
P_2
P_3
P_4
R_1
R_2
R_3
①
②
③
④
⑤
⑥
⑦
⑧
⑨
15 ft
20 ft
20 ft

FIG. 11

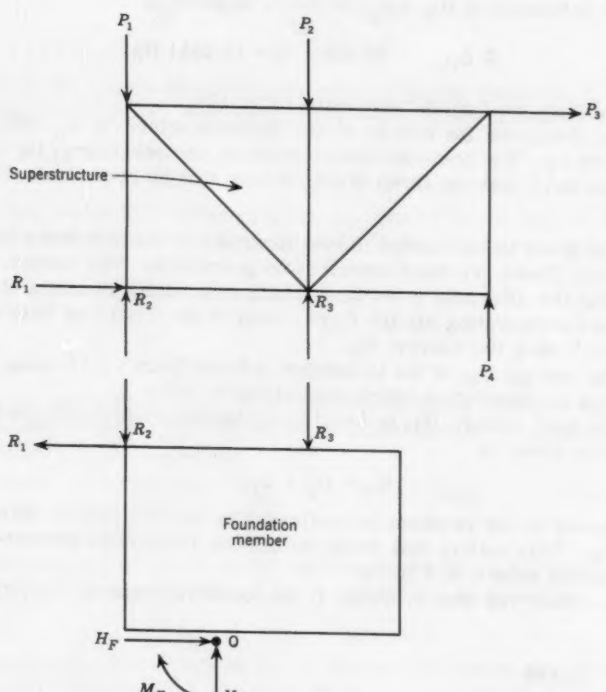
P_1
P_2
P_3
P_4
Superstructure
R_1
R_2
R_3
Foundation member
H_F
O
M_F
V_F

FIG. 12

TABLE 1.—BAR DATA AND STRESSES

Bar No.	L, in ft	A, in sq in.	L/A	Bar Stresses[a] for			
				P_1 = 1 kip	P_2 = 1 kip	P_3 = 1 kip	P_4 = 1 kip
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	20	11.44	1.748	0	0	+1	-1.33333
2	20	11.44	1.748	0	0	+1	-1.33333
3	20	11.44	1.748	0	0	+1	0
4	20	11.44	1.748	0	0	0	0
5	15	9.92	1.512	-1	0	+0.75	-1
6	15	9.92	1.512	0	-1	0	0
7	15	9.92	1.512	0	0	0	-1
8	25	8.72	2.867	0	0	-1.25	+1.66667
9	25	8.72	2.867	0	0	0	+1.66667

[a] A plus (+) sign denotes tension

TABLE 2.—REACTIONS

Reactions	Reactions for			
	P_1 = 1 kip	P_2 = 1 kip	P_3 = 1 kip	P_4 = 1 kip
(1)	(2)	(3)	(4)	(5)
R_1	0	0	-1	0
R_2	+1	0	-0.75	+1
R_3	0	+1	+0.75	-2

TABLE 3.—FOUNDATION FLEXIBILITIES

Deflection at	Due to		
	R_1 = 1 kip	R_2 = 1 kip	R_3 = 1 kip
(1)	(2)	(3)	(4)
R_1	0.24295	0	0
R_2	0	21.8081	16.2851
R_3	0	16.2851	18.1925

Note that the first row (and first column) corresponds to bar ①, etc. The term δ_k is a diagonal matrix. The flexibility of an axially loaded bar is given by the value of L/AE for the bar.

The matrix [a] given in Eq. 24, expressing the bar stresses in terms of the applied loads P_1, P_2, P_3 and P_4, is set up using the data in Table 1 and is

$$[a] = \begin{bmatrix} 0 & 0 & +1 & -1.33333 \\ 0 & 0 & +1 & -1.33333 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & +0.75 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1.25 & +1.66667 \\ 0 & 0 & 0 & +1.66667 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ (40)$$

Note in [a] the first column corresponds to P_1, etc., and the first row to bar ①, etc.

The value of U_S is given by

$$U_S = \frac{1}{2}\begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix}\begin{bmatrix}\delta_S\end{bmatrix}\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} \quad \ldots\ldots\ldots\ldots\ (41)$$

where

$$\begin{bmatrix}\delta_S\end{bmatrix} = [a]^T \begin{bmatrix}\delta_k\end{bmatrix} [a] \quad \ldots\ldots\ldots\ldots\ (42)$$

Performing the operations called for in Eq. 42, we find $\begin{bmatrix}\delta_S\end{bmatrix}$ given by

$$\begin{bmatrix}\delta_S\end{bmatrix} = \begin{bmatrix} +1.512 & 0 & -1.134 & +1.512 \\ 0 & +1.512 & 0 & 0 \\ -1.134 & 0 & +10.5742 & -11.7682 \\ +1.512 & 0 & -11.7682 & +25.1669 \end{bmatrix} \quad \ldots\ldots\ (43)$$

Note the symmetry of the result. Again the first row and first column refers to the load P_1, etc.

Step 2.—Using the data contained in Table 3, we can write the foundation flexibility matrix as

$$\begin{bmatrix}\delta_{kF}\end{bmatrix} = \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array}\begin{array}{c} \begin{array}{ccc} R_1 & R_2 & R_3 \end{array} \\ \begin{bmatrix} 0.24295 & 0 & 0 \\ 0 & 21.8081 & 16.2851 \\ 0 & 16.2851 & 18.1925 \end{bmatrix} \end{array} \quad \ldots\ldots\ldots\ (44)$$

The row and column designations are shown in (δ_{kF}) to emphasize that these flexibilities are in terms of R_1, R_2, and R_3. We need to express them in terms of P_1, P_2, P_3, and P_4, and for this purpose we set up a transformation matrix which expresses the R's in terms of the P's.

This is done using Table 2, to result in

$$[a_F] = \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array} \begin{array}{c} \begin{array}{cccc} P_1 & P_2 & P_3 & P_4 \end{array} \\ \begin{bmatrix} 0 & 0 & -1 & 0 \\ +1 & 0 & -0.75 & +1 \\ 0 & +1 & +0.75 & -2 \end{bmatrix} \end{array} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (45)$$

The row and column designations are shown to indicate the relationship of the R's to the P's.

Now the energy of the foundation, U_F, can be written as

$$U_F = \frac{1}{2} \left[R_1\ R_2\ R_3\right] \left[\delta_{kF}\right] \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} \quad \ldots\ldots\ldots\ldots\ldots \quad (46)$$

If we express the R's in terms of the P's by

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \left[a_F\right] \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (47)$$

then using Eq. 47 in Eq. 46, and noting Eq. 45, we have the energy in terms of P_1, P_2, and P_3, thus

$$U_F = \frac{1}{2} \left[P_1\ P_2\ P_3\right] \left[\delta_F\right] \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} \quad \ldots\ldots\ldots\ldots\ldots \quad (48)$$

where

$$\left[\delta_F\right] = \left[a_F\right]^T \left[\delta_{kF}\right]\left[a_F\right] \quad \ldots\ldots\ldots\ldots\ldots \quad (49)$$

Carrying out Eq. 49 we find

$$\left[\delta_F\right] = \begin{bmatrix} 21.8081 & 16.2851 & -4.14225 & -10.7621 \\ 16.2851 & 18.1925 & 1.43055 & -20.0999 \\ -4.14225 & 1.43055 & 4.17960 & -7.00335 \\ -10.7621 & -20.0999 & -7.00335 & 29.4377 \end{bmatrix} \quad \ldots\ldots \quad (50)$$

Step 3.—The total strain energy U_T is given by Eq. 38, and is equal to

$$U_T = \frac{1}{2} \left[P_1\ P_2\ P_3\ P_4\right] \left[\delta_P\right] \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} \quad \ldots\ldots\ldots\ldots\ldots \quad (51)$$

where

$$\left[\delta_P\right] = \left[\delta_S\right] + \left[\delta_F\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (52)$$

and using Eqs. 43 and 49 in Eq. 52, we have (at this point we reinsert E)

$$E\left[\delta_P\right] = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} P_1 & P_2 & P_3 & P_4 \end{array} \\ \begin{bmatrix} 23.3201 & 16.2851 & -5.2762 & -9.2501 \\ 16.2851 & 19.7045 & 1.4306 & -20.0999 \\ -5.2762 & 1.4306 & 14.7538 & -18.7716 \\ -9.2501 & -20.0999 & -18.7716 & 54.6046 \end{bmatrix} \end{array} \quad \ldots (53)$$

The term $E\left[\delta_P\right]$ given by Eq. 53 is our desired result. The P row and column designations are shown for purposes of identification.

PROBLEM 2

Given the three-span continuous beam on unyielding supports and under the loading shown in Fig. 13, it is desired to find all the reactions. In Fig. 13 A' is the area effective in resisting shear.

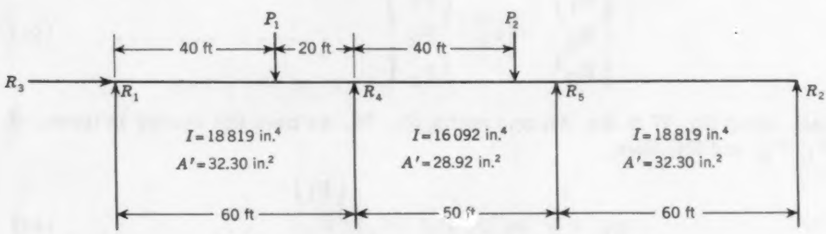

FIG. 13

This structure is statically indeterminate to the second degree. Furthermore, the horizontal reaction, R_3, is equal to zero and can be ignored in this problem.

The solution logically follows six steps:

(1) Selection of a statically determinate structure.

(2) Break-up of the structure into k parts.

(3) Calculation of the total energy using a matrix combining the flexibilities of the individual parts.

(4) Expression of the total energy in terms of the applied loads and redundants acting upon the statically determinate structure.

(5) Solution for the redundants.

(6) Evaluation of the reactions as functions of the applied loads.

Step 1.—The reactions R_1 and R_2 are selected as the redundants. The statically determinate structure, therefore, has R_1, R_2, P_1, and P_2 acting upon it.

Step 2.—Fig. 14 shows the break up of the structure into three parts lettered A through C. The required equilibrating forces are shown in Fig. 14. It should be noted that the structure is cut just to the left and right of the loads P_1 and P_2. Section A is clamped at its right end, section B at the cross section at R_4, and section C at the cross section at R_5.

If we consider the joints (points) upon which P_1 and P_2 act as pieces of structure, they do not contribute to the energy calculation at this step, since the strain energy of a point is equal to zero.

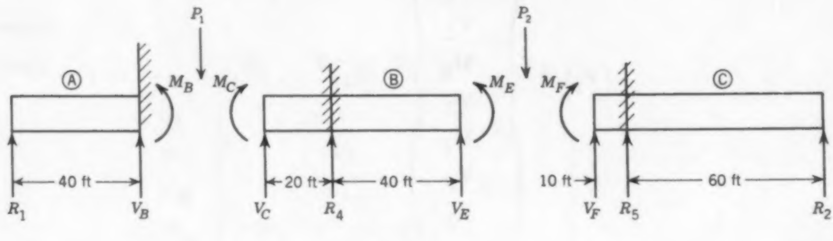

FIG. 14

Step 3.—The flexibilities for a simple cantilever acted upon at the free end by a moment, M, and a transverse force, V, are known to be

$$\delta_{MM} = \frac{L}{E\,I} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (54a)$$

$$\delta_{MV} = \delta_{VM} = \frac{L^2}{2\,E\,I} \quad \ldots\ldots\ldots\ldots\ldots \quad (54b)$$

and

$$\delta_{VV} = \frac{L^3}{3\,E\,I} + \frac{k\,L}{A'\,G} \quad \ldots\ldots\ldots\ldots\ldots \quad (54c)$$

where in the expression for δ_{VV} the effect of shear, $\frac{k\,L}{A'\,G}$, is included. If deflections, δ, are to be in feet, Eqs. 54 become (noting E is a constant and k is equal to 1.0 for I shapes),

$$E\,\delta_{MM} = 144\frac{L}{I} \quad \ldots\ldots\ldots\ldots\ldots \quad (55a)$$

$$E\,\delta_{MV} = E\,\delta_{VM} = 144 \times \frac{L^2}{2\,I} \quad \ldots\ldots\ldots\ldots \quad (55b)$$

and

$$E\,\delta_{VV} = 144 \times \frac{L^3}{3\,I} + 2.5 \times \frac{L}{A'} \quad \ldots\ldots\ldots\ldots \quad (55c)$$

In Table 4 are listed the flexibilities calculated in accordance with Eqs. 55. The term E will drop out of the calculations. Accordingly, we will ignore it for simplicity.

The total energy, U_T, can be written as

$$U_T = \frac{1}{2}\left[P_i\right]\left[\delta_{ij}\right]\left\{P_j\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (56)$$

where

$$\left\{P_j\right\} = \left\{\begin{array}{c} R_1 \\ M_C \\ V_C \\ M_E \\ V_E \\ M_F \\ V_F \\ R_2 \end{array}\right\} = \left[P_i\right]^T \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (57)$$

and, using Table 4,

$$\left[\delta_{ij}\right] = \begin{array}{cccccccc} 166.335 & & & & & & & \\ & 0.153037 & 1.53040 & & & & & \\ & 1.53040 & 21.953 & & & & & \\ & & & 0.357942 & 7.15884 & & & \\ & & & 7.15884 & 194.360 & & & \\ & & & & & 0.089485 & 0.44743 & \\ & & & & & 0.44743 & 3.8473 & \\ & & & & & & & 555.570 \end{array} \quad \ldots \quad (58)$$

The row (and column) designations correspond to the designations of the rows given for $\left\{P_j\right\}$ in Eq. 57.

TABLE 4.—FLEXIBILITY COEFFICIENTS

Section (1)	Forces (2)	E δ_{MM} (3)	E δ_{MV} (4)	E δ_{VV} (5)
A	R_1	–	–	166.335
B	M_C, V_C	0.153037	1.53040	21.953
	M_E, V_E	0.357942	7.15884	194.360
C	M_F, V_F	0.089485	0.44743	3.8473
	R_2	–	–	555.570

Step 4.—To express the total energy, U_T, in terms of the forces acting upon the statically determinate structure, we write the transformation

$$\left\{P_j\right\} = \left[a_{jy} \,\vdots\, a_{jB}\right] \left\{\begin{array}{c} P_y \\ \hline P_B \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (59)$$

Using the equations of static equilibrium and putting the sections together, we can obtain [a] by setting up the matrix product

$$[a] = \left[a_{jy} \,\vdots\, a_{jB}\right] = \left[b_{jk}\right]\left[C_{ky} \,\vdots\, C_{kB}\right] \qquad \text{(60)}$$

where

$$\left[b_{jk}\right] = \begin{array}{c} \\ R_1 \\ M_C \\ V_C \\ M_E \\ V_E \\ M_F \\ V_F \\ R_2 \end{array} \begin{array}{c} \begin{array}{ccccc} R_5 & R_1 & R_2 & P_1 & P_2 \end{array} \\ \left[\begin{array}{ccccc} & +1 & & & \\ & +40 & & & \\ & +1 & & -1 & \\ +10 & & +70 & & \\ +1 & & +1 & & -1 \\ +10 & & +70 & & \\ -1 & & -1 & & \\ & & +1 & & \end{array}\right] \end{array} \qquad \text{(61)}$$

and

$$\left[C_{ky} \,\vdots\, C_{kB}\right] = \begin{array}{c} \\ R_5 \\ R_1 \\ R_2 \\ P_1 \\ P_2 \end{array} \begin{array}{c} \begin{array}{cccc} R_1 & R_2 & P_1 & P_2 \end{array} \\ \left[\begin{array}{cccc} +1.2 & -2.2 & -0.4 & +0.8 \\ +1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{array}\right] \end{array} \qquad \text{(62)}$$

In both Eqs. 61 and 62 the row and column designations are shown for identification of elements.

Calculating [a] by Eq. 60 we can express the total energy, U_T, in terms of the forces as the statically determinate structure by using Eq. 60 in Eq. 59 and then Eq. 59 in Eq. 56 with the result

$$U_T = \frac{1}{2}\left[P_x \,\vdots\, P_A\right] \left[\begin{array}{c:c} \delta_{xy} & \delta_{xB} \\ \hdashline \delta_{Ay} & \delta_{AB} \end{array}\right] \left\{\begin{array}{c} P_y \\ \hdashline P_B \end{array}\right\} \qquad \text{(63)}$$

with

$$\left[\begin{array}{c:c} \delta_{xy} & \delta_{xB} \\ \hdashline \delta_{Ay} & \delta_{AB} \end{array}\right] = [a]^T \left[\delta_{ij}\right] [a]$$

$$= \begin{array}{c} \\ R_1 \\ R_2 \\ P_1 \\ P_2 \end{array} \begin{array}{c} \begin{array}{cccc} R_1 & R_2 & P_1 & P_2 \end{array} \\ \left[\begin{array}{cc:cc} +1098.72 & +262.23 & -264.21 & +42.953 \\ +262.23 & +1098.71 & -87.410 & +64.429 \\ \hdashline -264.21 & -87.410 & +82.301 & -14.318 \\ +42.953 & +64.429 & -14.318 & +10.237 \end{array}\right] \end{array} \qquad \text{(64)}$$

Step 5.—The redundants are given by

$$\{P_y\} = -\left[\delta_{xy}\right]^{-1}\left[\delta_{xB}\right]\{P_B\} = [D]\,\{P_B\} \quad \ldots\ldots\ldots \quad (65)$$

Carrying out the operations indicated in Eq. 65, we find

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = -\left[\delta_{xy}\right]^{-1}\left[\delta_{xB}\right]\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{bmatrix} 0.234865 & -0.026614 \\ 0.023501 & -0.052288 \end{bmatrix}\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \quad \ldots\ldots \quad (66)$$

Step 6.—To find the reactions in terms of P_1 and P_2, we form the matrix

$$[J] = \left[\frac{D}{I_2}\right] = \begin{array}{c} \\ R_1 \\ R_2 \\ P_1 \\ P_2 \end{array} \begin{array}{c} \begin{array}{cc} P_1 & \quad P_2 \end{array} \\ \begin{bmatrix} 0.234865 & -0.026614 \\ 0.023501 & -0.052288 \\ +1 & \\ & +1 \end{bmatrix} \end{array} \quad \ldots\ldots\ldots\ldots \quad (67)$$

We can express the reactions in terms of the redundants and the applied loads thus,

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_4 \\ R_5 \end{Bmatrix} = [g] \begin{Bmatrix} R_1 \\ R_2 \\ P_1 \\ P_2 \end{Bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (68)$$

with

$$[g] = \begin{array}{c} \\ R_1 \\ R_2 \\ R_4 \\ R_5 \end{array} \begin{array}{c} \begin{array}{cccc} R_1 & R_2 & P_1 & P_2 \end{array} \\ \begin{bmatrix} +1 & & & \\ & +1 & & \\ -2.2 & +1.2 & +1.4 & +0.2 \\ +1.2 & -2.2 & -0.4 & +0.8 \end{bmatrix} \end{array} \quad \ldots\ldots\ldots\ldots \quad (69)$$

Then our desired results are given by

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_4 \\ R_5 \end{Bmatrix} = [g]\,[J] \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (70)$$

Carrying out the indicated operations

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_4 \\ R_5 \end{Bmatrix} = \begin{bmatrix} +0.23486 & -0.02661 \\ +0.02350 & -0.05229 \\ +0.91150 & +0.19581 \\ -0.16986 & +0.88310 \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \qquad (71)$$

we have our actual answers.

This is as far as we will go in our discussion of the strain energy in terms of loads. We should, for completeness, consider the formulation of strain energy in terms of the displacements which define the deformed shape of the structure. In so doing we will develop the basic formulae and theory which, in its application, is known in the literature as the "method of stiffness" or the "stiffness" matrix approach.

SUPERPOSITION OF FORCES

Consider in Fig. 2, the axially loaded bar anchored at its left end. The cross sectional properties are constant.

We can write

$$\Delta_1 = \delta_{11} P_1 \qquad (72)$$

where δ_{11} is the flexibility coefficient. The inverse relationship is

$$P_1 = \frac{1}{\delta_{11}} \Delta_1 = k_{11} \Delta_1 \qquad (73)$$

where k_{11} is the stiffness coefficient. In the same manner that we define δ_{11} as the deflection at 1 due to a unit value of P_1, we can define k_{11} as the force at 1 due to a unit value of Δ_1.

We can generalize this concept using Fig. 1, and superimpose the forces, writing for any force, P_i,

$$P_i = k_{i1} \Delta_1 + k_{i2} \Delta_2 + \ldots k_{ij} \Delta_j \qquad (74)$$

where the k's are the stiffness coefficients. In matrix form Eq. 74 can be written as

$$\{P_i\} = [k_{ij}] \{\Delta_j\} \qquad (75)$$

where $[k_{ij}]$ is known as the stiffness matrix. The subscripts i and j have the same range. The equations given by Eq. 75 are analogous to those given by Eq. 3. Substitution of Eq. 3 into Eq. 73 requires

$$[k_{ij}] [\delta_{ji}] = [I] \qquad (76)$$

which indicates that $[\delta]$ is the inverse of $[k]$.

STRAIN ENERGY AS A FUNCTION OF DISPLACEMENTS

Fig. 15 is an outline of a simply supported pin-connected truss. The 13 horizontal and vertical arrows shown represent the joint displacements that

can take place when the truss deforms. Obviously, if these 13 deflections were known, we would know the deformed shape of the truss. These, then, are the 13 deflection parameters defining the strain energy of the truss.

Let us assume that forces act in the direction of these 13 deflections. The 13 arrows shown thus represent forces and displacements. If we call the deflections Δ_i and the forces P_i where i has the range 1 to 13, we can write for the total strain energy

$$U_T = \frac{1}{2} \sum_{i=1,\,2,\,--\,13} P_i \;\Delta_i \qquad (77)$$

or in matrix form

$$\left[U_T\right] = \frac{1}{2}\left[P_i\right]\left\{\Delta_i\right\} \qquad (78)$$

Using Eq. 75 in Eq. 78 we have as a result

$$\left[U_T\right] = \frac{1}{2}\left[\Delta_i\right]\left[k_{ij}\right]\left\{\Delta_j\right\} \qquad (79)$$

Note the similarity of Eq. 79 with Eq. 13.

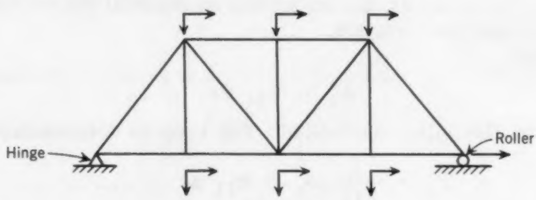

FIG. 15

In our study of strain energy as a function of the loads, we demonstrated that we could select any set of simple reactions, break up the structure into pieces, add the flexibilities of the pieces, and then obtain the total energy or sum the energies of the individual pieces. Space does not permit us to consider the same questions when strain energy is a function of displacements. Suffice it to say that we can do all these things except that in place of flexibilities we have stiffness and in place of loads we are dealing with displacements.

We can apply another of Castigliano's theorems, namely,

$$\frac{\partial U_T}{\partial \Delta_i} = P_i \qquad (80)$$

to the expression for the total strain energy given by Eq. 79 with the result

$$\left\{P_i\right\} = \left[k_{ij}\right]\left\{\Delta_j\right\} \qquad (81)$$

The $\left\{P_i\right\}$ are the forces acting at the points at which the deflections are considered acting. In an actual problem, no forces may be acting at some of these points or joints. To complete our formulation therefore, let us partition $\left\{P_i\right\}$ into two groups, as follows,

$$\left\{P_i\right\} = \left\{\frac{P_A}{o}\right\} \qquad (82)$$

where P_A represents those forces (or moments) that exist. Applying the same grouping to the Δ's we can rewrite Eq. 81 in the partitioned form

$$\left\{\begin{array}{c} P_A \\ \hdashline o \end{array}\right\} = \left[\begin{array}{c:c} k_{AB} & k_{An} \\ \hdashline k_{mB} & k_{mn} \end{array}\right] \left\{\begin{array}{c} \Delta_B \\ \hdashline \Delta_n \end{array}\right\} \qquad \text{. . . . . (83)}$$

in which the subscripts m and n refer to the points or joints where forces (or moments) are equal to zero.

We could continue and develop the detailed formulas used in the "method of stiffnesses" but as we are primarily concerned with the basic concept of energy, we shall stop at this point; there are, however, several papers (16, 19, 22) which deal with the theory and application of the "method of stiffnesses."

CONCLUSIONS

We have examined the concept of strain energy and have noted that it has a constant value for a given structure subjected to a set of forces in equilibrium. We have seen how we can obtain the total energy by breaking the structure up into convenient units, using arbitrary supports, and have shown the formulas for the solution of statically indeterminate structures. Two applications were then carried out. A similar investigation of strain energy expressed in terms of displacements was carried to a point where we could satisfy ourselves that earlier observations were applicable. We also discussed the inclusion of the foundation in our concept of the "whole" structure.

ACKNOWLEDGMENTS

The writer wishes to express his appreciation to the Fairchild Astrionics Division, Wyandanch, N. Y., for their material support and encouragement. He is particularly indebted to Douglas Craven and the staff of the Publications and Photography Departments of FAD for their assistance in the preparation of this paper.

NOTATIONS

Symbols and notation used in this paper are given below. Notation clearly defined in the text is not shown here:

$[\ \ ]$ = brackets denoting a rectangular matrix

$\{\ \ \}$ = column matrix

$[\ \ ]^T$ = transpose of a matrix

$[I_n]$ = unit matrix of order n

$[\delta]$ = flexibility matrix

$[k]$ = stiffness matrix

δ_{ij} = influence deflection coefficient; deflection in direction of $i^{\underline{th}}$ load due to a unit value of the $j^{\underline{th}}$ load

k_{ij} = influence stiffness coefficient; spring constant; force in the $i^{\underline{th}}$ direction due to unit value of deflection in $j^{\underline{th}}$ direction

Subscripts

x,y used to denote any redundant P_x, P_y. x and y have the same range of numbers, being used interchangeably.

m,n used to denote any applied load P_m, P_n. Same range of numbers or loads, being used interchangeably. Note n is used also for unit matrix.

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DISCUSSION

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A METHOD OF COMPUTATION FOR STRUCTURAL DYNAMICS[a]

Closure by Nathan M. Newmark

NATHAN M. NEWMARK,[1] F. ASCE.—The writer is pleased to have the contribution by Mr. Rosenblueth, which makes it possible to achieve in some cases greater accuracy with the numerical procedure. For undamped systems, and for linear behavior, the correction factors introduced by Mr. Rosenblueth may have some advantages. However, unpublished studies by the writer indicate that with damping the gain in accuracy with procedures of the type described may be illusory.

In either case, if one wishes to cut down the number of cycles of computation, one would attempt to use as large a time interval as possible. An upper limit to time interval is that which permits stability of the computations to be achieved. Unfortunately, methods which are of order of accuracy of h^4 are not usually as good for long time intervals as methods which are of the order of h^2, although they are substantially better when the time interval is short.

The word of caution described by Mr. Rosenblueth in the last paragraph of his discussion is an important one. Unless one takes account of the discontinuities in the input data, more refined methods may lead to inaccuracies rather than to increased accuracy.

[a] July, 1959, by Nathan M. Newmark.

[1] Head, Dept. of Civ. Engrg., Univ. of Illinois, Urbana, Ill.

ORTHOGONAL GRIDWORKS LOADED NORMALLY TO THEIR PLANES[a]

Discussion by Ming L. Pei

MING L. PEI,[4] M. ASCE.—The authors presented numerical solutions of a 4-beam gridwork taking into consideration the effects of torsional rigidities of the beams. The usual practice, when analyzing structures of this type, is to neglect all torsional rigidities of the beams, on the assumption that the torsional effect on deflections and moments is negligibly small. To test this hypothesis, the writer computed the deflections and moments of the same structure assuming zero torsional rigidities. The results are given below for comparison with the author's more accurate data.

From Table 1, it is seen that errors in deflections computed without consideration of torsional effects are rather large. These are not computational errors, but are discrepencies due to omission of the torsional effects. The beams in this example are circular bars. For more commonly used rectangular beams and structural steel shapes, the errors may not be as large, but will probably be large enough to warrant careful consideration. The author's statement that "the torsional rigidities of the bars should be taken into consideration in the solution of a gridwork" appears to be justified.

TABLE 1.—DEFLECTIONS AND ROTATIONS

Deflection	With torsion[a]	Without torsion[a]	Error, in percentage
θ_1	-0.0321	-0.04225	33.5
θ_2	0.0533	0.06713	27.7
θ_3	-0.0208	-0.02546	22.0
θ_4	0.0275	0.03183	15.3
δ_1	0.0700	0.07542	7.7
δ_2	0.0300	0.03395	11.3
δ_4	0.0202	0.02334	15.4

[a] Multiply all values by $P\ L^2/2\ E\ I$.

The extremely large error in M_{2d} deserves comment. The bending moment is computed from deflections θ_2 and δ_2 by Eq. 10(f). The errors in θ_2 and δ_2 are only 27.7% and 11.3%, respectively. This example demonstrates that even if the values of deflections (or rotations) have been computed fairly

[a] January, 1960, by Ignacio Martin A. M. ASCE, and José Hernandez.

[4] Assoc. Prof., City College of New York, N. Y.

accurately, there is no guarantee that the values of bending moments obtained from these approximate deflections will have comparable accuracy.

The authors formulated the illustrative problem in a set of 7 simultaneous equations, Eq. 15(a) to 15(g). It should be pointed out that these equations should be listed in proper order as follows:

$$4.72\,\theta_1 + \theta_2 - 0.36\,\theta_3 + 0 + 0 + 3\,\frac{\Delta_2}{L} + 0 = 0 \qquad \ldots(15a)$$

$$\theta_1 + 4.72\,\theta_2 + 0 - 0.36\,\theta_4 - 3\,\frac{\Delta_1}{L} + 0 + 0 = 0 \qquad \ldots(15c)$$

$$-0.36\,\theta_1 + 0 + 4.72\,\theta_3 + \theta_4 + 0 + 0 + 3\,\frac{\Delta_4}{L} = 0 \qquad \ldots(15d)$$

$$0 - 0.36\,\theta_2 + \theta_3 + 4.76\,\theta_4 + 0 - 3\,\frac{\Delta_2}{L} + 0 = 0 \qquad \ldots(15f)$$

$$0 - 6\,\theta_2 + 0 + 0 + 24\,\frac{\Delta_1}{L} - 12\,\frac{\Delta_2}{L} + 0 = \frac{PL^2}{2EI} \qquad \ldots(15b)$$

$$3\,\theta_1 + 0 + 0 - 3\,\theta_4 - 6\,\frac{\Delta_1}{L} + 24\,\frac{\Delta_2}{L} - 6\,\frac{\Delta_4}{L} = 0 \qquad \ldots(15c)$$

$$0 + 0 + 6\,\theta_3 + 0 + 0 - 12\,\frac{\Delta_2}{L} + 24\,\frac{\Delta_4}{L} = 0 \qquad \ldots(15g)$$

Of course, a set of equations has only one set of exact solutions, irrespective of which equation is listed as the first equation. When the number of equations

TABLE 2.—BENDING MOMENTS AND TORQUES

Moment	With torsion[a]	Without torsion[a]	Error, in percentage
M_{a1}	0.1779	0.17401	2.3
M_{1a}	0.1458	0.13176	9.6
M_{h3}	0.0692	0.07639	10.4
M_{3h}	0.0484	0.05093	5.2
M_{d2}	-0.0367	-0.03472	5.4
M_{2d}	0.0166	0.03241	95.5
M_{e4}	-0.0331	-0.03819	15.5
M_{4e}	-0.0056	-0.00636	13.5
M_{12}	-0.1309	-0.14178	8.3
M_{21}	-0.0450	-0.03240	27.5
M_{34}	-0.0435	-0.05092	17.0
M_{43}	0.0048	0.00637	32.8

[a] Multiply all values by P L.

is small, it is rarely necessary to worry about the proper listing of the equations. However, when dealing with large sets of equations, whether by hand or computers, it is essential that the equations be arranged in their proper order for efficient computation.

The authors are to be congratulated for using the more accurate analysis in their engineering design.

CONTINUOUS GIRDER BRIDGE WITH VARIABLE MOMENT OF INERTIA[a]

Discussion by Valerian Leontovich

VALERIAN LEONTOVICH,[8] M. ASCE.—The author of the paper is to be commended for bringing this subject to the attention of engineers and for his presentation of "ready-to-use" solutions for determining the moments in continuous girders of parabolic shape. The presentation of the tabulated constants is also a valuable contribution to the engineer's notebook.

This writer, however, is at variance with the author's opinion, that the methods of analysis of continuous girders of variable moment of inertia, presently available, are complex, tedious and time consuming. Much depends, of course, on the judicious selection of the method by the designer.

Certainly, there are methods that provide simple, direct, and precise solutions to the aforestated problems. Furthermore, some of the methods permit formulation of analytical solution in general terms, thereby providing means for solution of problems with members of any shape and cross-sectional variation.

Of recent development in this field is the writer's method based on the Concept of Elastic Parameters,[9] which affords an opportunity to perform the solution of analysis in the above stated form for frames, arches, and continuous beams of variable cross section. The utility of the Concept in application to analysis of frames and arches has been demonstrated previously,[9] and it is apparent that its application to the analysis of continuous beams is the most simple and elementary procedure. Furthermore, numerical values of necessary elastic parameters and load constants have already been developed and are available.[10]

The solution of the problems presented in the paper are carried out below by the writer's method in order to demonstrate its inherent simplicity and preciseness.

Symmetrical Two-Span Continuous Girder Bridge With Variable Moment of Inertia.—Based on the relationships presented by the "Concept of Elastic Parameters" one may write a fundamental three-moment equation for any two adjacent spans of a continuous girder of variable moment of inertia. Thus,

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[a] January, 1960, by Sabri Sami.

[8] Senior Design Engr., Bechtal Corp., San Francisco, Calif.

[9] Proceedings, Amer. Concrete Inst., Vol. 54, May, 1958.

[10] "Frames and Arches," by Valerian Leontovich, McGraw-Hill Book Co., Inc., New York, 1959.

for example, for an arbitrary two-span continuous girder (AB,BC), the equation would be

$$\frac{1}{E\,I^{\circ}_{B-C}}\left[\frac{M_A\,L_{A-B}\,\beta_{AB}}{\phi}+M_B\left(\frac{L_{A-B}\,\alpha_{BA}}{\phi}+L_{B-C}\,\alpha_{BC}\right)+\right.$$

$$\left.M_C\,L_{B-C}\,\beta_{BC}\right]=-\frac{1}{E\,I^{\circ}_{BC}}\left[\frac{R_{BA}\,W_{A-B}\,L^2_{A-B}}{\phi}+R_{BC}\,W_{B-C}\,L^2_{B-C}\right]\quad\ldots(51)$$

in which E is the modulus of elastisity, I°_n refers to the minimum moment of inertia of a member's cross section about its neutral axis; (the subscript identifies the member), L_n denotes the span of the member defined by the subscript, M_n is the bending moment at the section defined by the subscript, R_n indicates the load constant of the particular end of the member (the end of the member is defined by the subscript), W_n is the total load on the member defined by the subscript, α_n, and β_n are elastic parameters of the individual member for the end of the member defined by subscript n, and ϕ is the ratio $\dfrac{I^{\circ}_{AB}}{I^{\circ}_{BC}}$.

Applying Eq. 51 to a two span continuous girder with simply supported ends, and realizing that moments at the first and third supports are zero, the equation defining moment over the central support reduces to the form

$$M_B=-\frac{W_{A-B}\,L_{A-B}\,R_{BA}+W_{B-C}\,L_{B-C}\,R_{BC}}{\alpha_{BA}+\alpha_{BC}}\quad\ldots\ldots\ldots(52)$$

Thus, by writing one equation and reducing it, the analytical part of the problem is solved for any symmetrical or unsymmetrical two-span girder and for any loading on one or both spans. Furthermore, the equation is perfectly general and applicable to members with parabolic or straight haunches or where either one or both of the members are prismatic.

In the case of the symmetrical two-span girder, which is considered by the author in his paper, the spans are equal, I° is the same for both spans and, consequently, $\alpha_{BA}=\alpha_{BC}$. When uniform and equal loads on both spans are considered, it is apparent that $W_{A-B}=W_{B-C}$ and $R_{BA}=R_{BC}$. After cancellations and reduction, Eq. 52 takes the form

$$M_B=-\frac{W_{A-B}\,L_{A-B}\,R_{BA}}{\alpha_{BA}}\quad\ldots\ldots\ldots\ldots\ldots(53)$$

Substituting constants R_{BA} and α_{BA} from tables and charts the problem is solved numerically. For any other shape of this symmetrical girder, only numerical values of α_{BA} and R_{BA} need be replaced.

Illustrative Example.—Consider the two-span symmetrical girder described in the author's paper. The ratio of the min to max moment of inertia of the member's sections is

$$\frac{(\min d)^3}{(\max d)^3}=0.1457$$

For this ratio, member's shape and load, the parameters and load constants, from the charts,[10] are $\alpha=1.28$ and $R=0.227$. The moment at central support is

$$M=-\frac{0.227\;W\,S}{1.28}=-0.1773\;w\,S^2$$

The center reaction is 100 (60 - 60 × 2 × 0.1773) = 8,128 lb, which is in very good agreement with 8,125 lb obtained by the author. If for some reason it is desired to change the shape or cross-sectional dimensions of the members, it is only necessary to select new constants α_{BA} and R_{BA} from tables and insert them into Eq. 53 to yield a new solution.

Symmetrical Three-Span Continuous Girder Bridge With Variable Moment of Inertia.—Based on the Concept of Elastic Parameters for any three-span symmetrical continuous girder A-B, B-C, C-D, of variable moment of inertia, but whose modulus of elasticity, E, is assumed the same for all members of the girder, two three-moment equations may be written. These are

$$\frac{M_A\, L_{A-B}\, \beta_{AB}}{\phi} + M_B\left(\frac{L_{A-B}\, \alpha_{BA}}{\phi} + L_{B-C}\, \alpha_{BC}\right) + M_C\, L_{B-C}\, \beta_{BC} = -\left[\frac{W_{A-B}\, L^2_{A-B}\, R_{BA}}{\phi} + W_{B-C}\, L^2_{B-C}\, R_{BC}\right] \qquad (54)$$

and

$$M_B\, L_{B-C}\, \beta_{BC} + M_C\left(L_{B-C}\, \alpha_{CB} + \frac{L_{C-D}\, \alpha_{CD}}{\phi}\right) + \frac{M_D\, L_{C-D}\, \beta_{CD}}{\phi} = -\left[W_{B-C}\, L^2_{B-C}\, R_{CB} + \frac{W_{C-D}\, L^2_{C-D}\, R_{CD}}{\phi}\right] \qquad (55)$$

Solving these two equations simultaneously, the solution of the analysis is obtained in general terms for a symmetrical continuous girder of any span, for variable moment of inertia, and for a variety of loads on the girder. Thus, for example, for moment M_C the expressions are as follows:

Left span loaded;

$$M_C = -\,W_{A-B}\, L_{A-B}\, R_{BA}\, N \qquad (56)$$

Center span loaded;

$$M_C = W_{B-C}\, L_{B-C}\, k\left(R_{CB}\, Q - R_{BC}\, N\right) \qquad (57)$$

Right span loaded;

$$M_C = W_{C-D}\, L_{A-B}\, R_{CD}\, Q \qquad (58)$$

in which

$$k = \frac{\min I_{A-B}}{\min I_{B-C}}\; \frac{L_{B-C}}{L_{A-B}} \qquad (59)$$

$$N = \frac{k\, \beta_{BC}}{k^2\, \beta^2_{BC} - \left(\alpha_{BA} + k\, \alpha_{BC}\right)^2} \qquad (60)$$

and

$$Q = \frac{\alpha_{BA} + k\, \alpha_{BC}}{k^2\, \beta^2_{BC} - \left(\alpha_{BA} + k\, \alpha_{BC}\right)^2} \qquad (61)$$

Illustrative Example.—The author's Example 2, is taken to demonstrate the solution by the writer's method. The data is as follows:

A three-span continuous girder has 182-ft center span and 130-ft side spans. The cross sections of members are varied in accordance with the parabolic

law defined by Fig. 6. The depth of the members at the outer supports and at the center line of the girder is assumed to be unity, while the depth of the members over the inner supports is assumed as three units.

The solution then is as follows: From charts[10] the following elastic parameters of the members are obtained: for member A-B, α_{BA} = 0.59; for member B-C, α_{BC} = 1.36 and β_{BC} = 1.06.

Inserting the dimensional properties of the girder and the elastic parameters into Eqs. 59, 60, and 61, we obtained k = 1.4, N = -0.37, and Q = -0.622.

Substituting numerical values into Eqs. 56, 57, and 58, the magnitude of the moment over support "C" for any loading condition on the three-span continuous girder may be determined.

Thus for example, with a total load W_{B-C} acting uniformly on the center span the load constants (from the same source as stated previously are R_{BC} = R_{CB} = 0.27. By substituting in Eq. 57, the numerical value of moment M_C is obtained.

$$M_C = W_{B-C}\ L_{B-C}\ k\ R_{BC}\ (Q-N) =$$

$$W_{B-C}\ L_{B-C}\ 1.4\ 0.27\ (-0.622 + 0.37) = -0.0953\ W_{B-C}\ L_{B-C}$$

or in terms of span L_{A-B} and load w per linear foot of span,

$$M_C = -0.186\ w\ L^2_{A-B}$$

The above problem has been worked to slide rule accuracy. The result however, is in very good agreement with the value obtained by the author, the deviation being about 1%.

It is believed that the preceding figures will demonstrate that by the use of the writer's Concept of Elastic Parameters, problems involving continuous beams with variable section may be solved precisely, in a short time, and with the use of ordinary office equipment only.

The presentation of these comments should not be construed as intending to detract from the value of Mr. Sami's valuable contribution, but rather to present further approaches, only recently available for the solution of similar problems.

PROPERTIES OF STEEL AND CONCRETE AND THE BEHAVIOR OF STRUCTURES[a]

Discussion by Henry J. Cowan and Oliver G. Julian

HENRY J. COWAN,[7] M. ASCE.—Mr. Winter is to be congratulated on an outstandingly lucid paper. The writer agrees with most of the author's comments. He would, however, like to correct a small error in reference to his own work. After discussion of this work, the author states: These various approaches have in common the fact that they attempt to relate the strength of concrete under continued stress to a single property, usually the cylinder strength. . ." The criterion for failure under combined stresses used by the writer, as the author states correctly, in an earlier passage, is a dual criterion, in which the primary tension failure is governed by the maximum stress theory, and the primary compression failure by the Mohr theory (with the Coulomb straight line criterion serving as a good approximation within the range considered.) The compressive strength and the tensile strength of the concrete are, therefore, determined by two independent series of tests.

In the writer's investigation, referred to by Mr. Winter, it was the aim to keep the quality of the concrete uniform throughout the investigation, to determine the compressive strength of the concrete for each reinforced test specimen from concurrent cylinder tests, and to determine the average tensile strength for the entire series from torsion tests on plain concrete. In actual fact, there was some variation in concrete quality due to the fact that the program of tests was undertaken at a time of severe fuel restrictions in England, so that it was impossible to heat the laboratory in winter, on weekends, and holidays and this produced a seasonal variation of strength. In consequence, the tensile strength for the individual specimens was derived from the average strength of the series and from the individual cylinder tests by assuming that the tensile strength was proportional to $\sqrt[3]{(\text{compressive strength})^2}$.

While the approximate character of this correction may have affected the interpretation of the data, it in no way alters the failure criterion, which is based on two independent quantities, the tensile strength and the compressive strength.

OLIVER G. JULIAN,[8] F. ASCE.—The author has presented an excellent, cogent summary of currently developed technical features of structural engineering as applied to steel and concrete. It is noted that it was presented to introduce a symposium on plasticity in structural design. The intent of this discussion is to confirm and emphasize certain points that the writer considers of particular importance.

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[a] February, 1960, by George Winter.

[7] Prof. and Head, Dept. of Architectural Science, Univ. of Sydney, Australia.

[8] Head of Structural Div., Jackson and Moreland, Inc. Engrs., Boston, Mass.

Under the section on the yield point of steel, the author states that "it is unlikely that it will ever be possible to deal with safety entirely on a statistical - probability basis." The writer has considerable sympathy with this point of view, with emphasis on the word "entirely." However, he feels compelled to add that apparently the greatest impediment to the use of such an approach is lack of knowledge by many regarding the elements of the mathematics involved, and reluctance to admit or recognize, that no matter how conservatively a structure is designed, there is in the nature of things some probability of its failure. This probability may be extremely small, say 10^{-4} or 10^{-6}, however, it is finite (not zero)(5).

It should be borne in mind that in addition to the variations in the quality of material (such as yield point) mentioned by the author, the strength of a structure depends on variations in quality of workmanship (accuracy of fabrication and erection), errors in design, and perhaps other variations. In probability studies, the combined effect of such variations is usually measured by the vectorial sum of the individual coefficients of variation. For example, if the individual coefficients are as follows: material as delivered on the job 0.124; workmanship on job 0.050; and errors in design 0.050; the combined coefficient of variation is $\sqrt{(.124)^2 + (.05)^2 + (.05)^2} = .143$.

The writer wholeheartedly endorses the author's statement to the effect that the revolution now underway in structural engineering may be aptly characterized as the strength concept versus the stress concept. However, although in many cases the value of intensity of stress may be of no practical importance, in many cases it is of prime importance because (1) it may in some cases govern the usefulness of the structure as measured by deformations and deflections, and (2) in other cases it may govern its strength with respect to collapse or unduly shorten its useful life. As a matter of fact, we have been using the principles of so called "plastic design" for many decades while giving lip service to the theory of elasticity.

The writer wishes to emphasize that we need both concepts. One cannot rationally design structures using the strength concept without having a working knowledge of the stress concept including what is commonly miscalled "strength of materials" and the theories of elasticity and elastic stability. In addition, it is of course necessary to know the pertinent values of the properties of the materials and their probable variations. We should be cautious not to oversimplify procedures for plastic design to a degree that will, in many cases, produce structures which, although safe with respect to collapse, will be unserviceable on account of excessive deformations and deflections or be unsightly, and, in many cases, may be dangerous or have an unduly limited useful life. As pointed out and illustrated by the author, "it may be necessary to surround plastic design with more refined safeguards than are necessary for conventional procedures."

The writer is well acquainted with the background of Figs. 7(a) and 7(b). For more detailed and pertinent information than is given on these Figs., reference may be made to the writer's Figs. 3 and 4, Table I and the accompanying text as referenced by the author (5). It appears desirable to state here that the coefficients of variation associated with Figs. 7(a) and 7(b) are 0.104 and 0.263 respectively. It will be noted that the mean strength (5.2 ksi) of the concrete represented by Fig. 7(a) is 1.4 ksi, that is, 37% greater than 3.8 ksi which is designated as the specified minimum strength. Such a margin is considered unwarranted. The writer's practice is to aim the mean strength of concrete as placed on the job (not of laboratory trial mixes) 18% higher

than the minimum specified. Then, if the coefficient of variations closely approximates that associated with Fig. 7(a), at least 90% of the test cylinders may be expected to exceed the minimum specified strength.

One of the intangibles associated with concrete control is cooperation by the owner. Good concrete control is relatively expensive. It may approximate \$3.00 per cu yd on a typical power house job where the placing of concrete is of necessity intermittent. This includes services of two control engineers, one at the proportioning plant and one at the location where the concrete is placed, and the services of a commercial laboratory to make and report the results of tests. It is not unusual to consider the cost of adequate control excessive and not worthwhile. The notion that concrete is a "fool-proof" material, which can be proportioned, mixed, transported, placed, and not cured without engineering supervision supplemented by tests of all materials and the concrete as placed, is all too prevalent.

Regarding the tensile strength of concrete, values of which are quoted by the author as fractions varying between 1/8 and 1/30 of the cylinder strengths, it appears to the writer that a far better gage for this important property is

$$f'_t = k\sqrt{f'_c} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

in which f'_t is the tensile strength, f'_c refers to the compressive cylinder strength (both strengths being in pounds per square inch), and k is a parameter having the dimensions $\text{lbs}^{\frac{1}{2}}\ \text{in.}^{-1}$.

From such test data as he has reviewed, the writer judges that a conservative value for k is 3. However, it apparently varies between wide limits from $2\frac{1}{2}$ to 5, dependent upon conditions. The author's statement to the effect that lack of information regarding the tensile strength of concrete seriously hampers attempts toward deeper understanding of concrete performance, is very much to the point. It is hoped that it will encourage research regarding this important property.

An expression similar in form to Eq. 8 appears suitable and desirable for bond strength. In this case, the writer is not in a position to even suggest a value for k, especially for the larger sized bars such as 14S and 18S.

An important property of concrete not covered explicitly by the author is Young's modulus, E_c, and its variations with time, exposure, and loading conditions. This parameter is of particular importance when considering buckling problems and in computing deflections. It is currently customary to take the value of E_c for short time loading as 1,000 times the compressive cylinder strength f'_c. Although this value may be a fair approximation in case $f'_c \approx$ 2,500 psi, it does not appear to be suitable for use in connection with the higher strength concretes. A preferable expression suggested to the writer by I. M. Viest, M. ASCE is

$$E_c \approx 50{,}000\sqrt{f'_c} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

in which E_c and f'_c are in psi and the constant has the dimensions $\text{lbs}^{\frac{1}{2}}\ \text{in.}^{-1}$. For a long time loading, this value should be reduced between 10% and 75%, dependent upon conditions, in order to obtain the effective values.

If the concrete is maintained in a continuously moist condition, there may be considerable increase in the value of f'_c which will be reflected in the value of E_c. However, this will probably be more than offset by the effect of creep so that a reduction of 10% will be in order. On the other hand, if the concrete is dried out, shrinkage as well as creep must be considered. There will be a negligible increase in the value of f'_c and a reduction of 75% will be in order.

It is noted that this latter reduction is in line with the greatest reduction noted by the author in discussing computations of deflections. The reductions stated above are for members subjected to continuous application of load. They should not be used in connection with that portion of the live loading which is applied only for short intervals.

The value of E_c suitable for use in computing the critical loads for combination columns, such as those consisting of steel pipe filled with concrete, is indeed debatable. The value currently employed by the writer is 440 f'_c. If f'_c = 4,200 psi this implies a 43% reduction from the value given above for short time loading.

The suspicion expressed by the author, that the descending branch of the stress-strain curve for concrete is caused by progressive irreversible microcracking is most plausible. It apparently solves an enigma of long standing, namely, how is it possible, in sound material, for the stress to be other than maximum at the location of maximum strain. As indicated by the author, it is questionable if the descending branch of the stress-strain curve should be considered as effective. If it is not so considered, the compressive strain of the concrete, according to Fig. 8, should be limited to roughly 0.002, which approximates the yield strain of steel having a yield strength of 60 ksi. The use of steels of higher yield strength may cause greater strains than it is possible to develop in the concrete and, therefore, as pointed out by the author, put a limit on their efficiency for compressive reinforcement. The advisability of their use for tensile reinforcement is also somewhat questionable as it may lead to undesirable deflections.

In the sections on failure under combined stresses and shear strength of beams the author presents an especially noteworthy and clear general view regarding the shear strength of beams. The quotation from Bresler and Pister (22) reading,

> "that strength of concrete is a function of the state of stress and cannot be predicted by limitations of tensile, compressive and shearing stresses independently of each other."

although axiomatic (at least to those acquainted with elementary strength of materials), cannot be emphasized too strongly.

It is indeed unfortunate that although strength theories such as Coulomb's and Mohr's apparently apply failry well to concrete subjected to compression, they do not apply to concrete subjected to tension for which Rankine's theory appears to apply. This seriously complicates problems involving combined stress such as the shear strength of concrete beams. Other complexities are introduced by shrinkage and temperature effects which are all too often neglected.

In spite of these complexities we have for many decades been designing beams by limiting the computed shearing stress (as a measure of diagonal tension) to a fractional part of f'_c and totally disregarding the quotation previously given (22). The fractional parts of f'_c specified have, in some cases, proved to be other than lower limits of the involved strength.

On account of the sudden nature of shear failures, the complexity of the problem, the large variation in reported test results, and the small relative differences in costs (when considered as a percentage of the cost of the finished concrete frame), the writer believes we should be sure to be on the

conservative side with respect to shear strength of concrete structures. Salient points regarding the writer's practice are:

1. Limit the unit working shearing stress v_c for members without web reinforcing to $1\sqrt{f'_c}$. This expression is the same as that given previously for the ultimate strength of concrete in tension except for the inclusion of a "factor of ignorance" of 3.

2. If the unit working shearing stress exceeds $1\sqrt{f'_c}$ assume that the entire shearing force at a section is resisted by adequate web reinforcing stressed at not more than 20,000 psi. (This for new billet-steel with a minimum specified yield strength of 40,000 psi).

DYNAMIC EFFECTS OF EARTHQUAKES[a]

Discussion by Clarence J. Derrick

CLARENCE J. DERRICK,[6] F. ASCE.—The author's brief justification of the SEAOC proposal,[7] on the ground of consonance with the essentials of dynamic theory leaves a number of unanswered questions.

One of these is how the extremely simple expression for the basic seismic factor, "C" = $\frac{0.05}{\sqrt[3]{T}}$, which employs a single criterion, can express, simultaneously: (a) S_v which varies principally with "damping" (the variation with T being directly with the first power), (b) W_n which is a function of the distribution of mass and elasticity along the height of the structure, and not of T, (c) The "influence" of higher mode response," which, as Clough showed,[8] varies more with "damping" than with the pattern of the spectrum; and (d) "Inelastic action," which Jacobsen concluded,[9] may be represented, for "amplitudes of earthquake interest" by the results of elastic analysis, employing the concept of "equivalent viscous damping."

The simplicity of C is more perplexing when it is considered that the SEAOC proposal ignores "damping," distribution of elasticity along the height of the structure, and the influence of foundation conditions.

The author's comparison of V_n, the base shear deduced from dynamic theory with V, the base shear prescribed by the SEAOC proposal, seems to imply direct proportionality, if not approximate identity. It is evident, from the constitution of the two expressions that such implication is unreasonable. The dynamic expression is influenced by both damping and distribution of elasticity along the height of the structure. These two variables, however, are not recognized by the SEAOC proposal.

This non-recognition of damping, by the SEAOC proposal, makes difficult the rationalization of V. If some reasonable value of damping, such as 20% of "critical" is selected, the ratio of V_n/V may be established, for a given spectrum, at various periods. This ratio is not constant, being higher at shorter periods than at longer periods, the variation being from as high as 6.0 to as low as 2.0, depending upon the spectrum employed. At given periods, the ratio may be made to vary by substituting a spectrum deduced from records made on soft ground for those computed from recordings on hard ground. Since the SEAOC proposal does not recognize the influence of variations in foundation conditions, such comparisons are inconclusive.

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[a] April, 1960, Ray W. Clough.

[6] Cons. Structural Engr., Los Angeles, Calif.

[7] "Recommended Lateral Force Requirements," Structural Engrs. Assn. of Calif., July, 1959.

[8] "On the Importance of Higher Modes of Vibration in the Earthquake Response of a Tall Building," by R. W. Clough, Bull. Seis. Soc. of Amer., Vol. 45, No. 4, October, 1955.

[9] "Frictional Effects in Composite Structures Subjected to Earthquake Vibrations," by L. S. Jacobsen, Dept. of Mech. Engrg., Stanford Univ., March 9, 1959, pp. 1-2.

The author places considerable emphasis upon the presumed beneficial effect of "inelastic action" without offering any adequate explanation. At present, the effect of "inelastic action" in complex multi-story buildings has not been explored, either analytically or experimentally. That such action occurs seems probable but as Jacobsen concludes, the contribution seems capable of anticipation by conventional elastic analysis, using the concept of equivalent viscous damping. The author's use of this factor as a basis for justification appears speculative. Since the SEAOC proposal contains no quantitative criteria for either "damping" or "inelastic action," such justification is not reasonable.

This adverse criticism applies particularly to the factor K, which the Seismology Committee of SEAOC, describes as a "bonus consideration for certain types of construction." From the explanation given in the "Introduction" to the published "Recommendations," the values of K appear to have been derived empirically. There is no suggestion of any quantitative use of "inelastic action," particularly since the values are not affected by either the nature of materials used or the manner of their interconnection.

The value of V, in the SEAOC proposal, is prescribed by the actual weight and a "seismic factor," the product, K C. Since K is established by the general type of construction, empirically, if not arbitrarily, much depends upon the precision with which C is evaluated.

The sole criterion of C is the reciprocal of the cube root of T, which is described as the fundamental free period of the structure. The precise evaluation of this sole criterion is critical. The author's statement that the SEAOC proposals "provide a rational basis for design of earthquake-resistant structures" implies satisfaction with the prescriptions for establishing the value of T.

Except for one specific type of construction, the proposal permits the designer to submit "properly substantiated technical data for establishing the period T." In lieu of such submission, the value of T may be "determined" by use of a formula,

$$T = \frac{0.05\ H}{\sqrt{D}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

in which H is the height and D the breadth of the building, at the base, in the direction considered.

This formula does not contain, either expressly or by implication, the basic criterion for free period, the "mass/elasticity" ratio. Hence, it is not rational. Moreover, the criterial ratio, $H/\sqrt{D}$, is associated with free period in "flexure", the influence of "shear" deflection disregarded. For a specific type of structure—for example, with a "moment-resisting space frame" which, acting without interference from more rigid elements is capable of resisting 100% of the required lateral forces (defined by K and C)—the designer is required to assume that T = 0.10 N, where N is the number of stories. This formula derives its dubious validity from consideration of "shear" deflection alone, the "flexural" deflection being disregarded.

This leads to some curious conclusions. For such a structure, the free period is presumed to be the same in both principal directions, regardless of the orientation of columns or direction of principal floor members. Moreover, since this type of construction, regardless of materials used or the manner of their interconnection, enjoys the lowest value of K, the value of V, identical in

both principal directions, is $V = \frac{0.072}{\sqrt[3]{N}} W$. This indicates that the base shear is solely a function of total weight and number of stories.

Since a competent structural engineer can evaluate T, with design accuracy, from the physical peculiarities of the proposed structure and he is obliged, by Sub-section (e) to perform the most difficult of such evaluation for another purpose, there appears to be only one reason for use of a formula to "determine" the fundamental free period. It must be inferred that, in the SEAOC proposal, T is a device for controlling C and other prescriptions, within limits deemed proper. This is not unjustifiable but it is empirical and may be opinionative. It is not consonant with dynamic theory.

From the standpoint of practical design, there is some justification for establishing V, the design base shear, as a fraction of V_n, the actual maximum dynamic base shear. This was explained by Housner,[10] and is a doctrine generally accepted by structural engineers in Southern California, including the writer. This doctrine is simple, merely that the designer should consider two levels of protection rath^r than one. The first is minimization, if not elimination, of all damage, "non-structural" as well as structural, likely to be produced by earthquakes of "moderate" local intensity. The second is absolute protection against collapse, even at the cost of some structural failure, during an earthquake of maximum local violence, or, as Housner puts it, "extremely strong ground motion."

The "spectra" provide a simple solution. Except for variations along the distribution of T due to response of different types of foundation columns, "moderate" earthquakes have about half the damage potential of "extremely strong" earthquakes. The ratio of $\frac{V_n}{V}$ becomes 2.0.

This conclusion appears to have been reached, intuitively and empirically by code-makers, in so far as structural steel construction is concerned. By coincidence, the elastic limit of steel is about half the ultimate strength and, by keeping the "allowable working stress" below the elastic limit, there is provided a "factor-of-safety" at the "near-collapse stress" level. Hence, a steel frame building that, as Housner puts it, will "ensure that no damage result(ed) from moderate elastic vibrations" will probably possess "an adequate factor of safety against collapse in event of extremely strong ground motion", for example, by taking advantage of "inelastic action" and permitting some minor structural damage. This empirical solution does nto apply, necessarily, to all materials of construction.

In practice, the adequacy of such design depends upon the ratio, $\frac{V_n}{V}$.

If this ratio exceeds 2.0, the "factor-of-safety" against collapse is reduced. If it exceeds 2.5, for example $\frac{60{,}000}{24{,}000}$, there is no "factor-of-safety" in a structural steel frame except that provided by the still unproven contribution by "inelastic action." For ratios above 3.0, the "equivalent viscous damping ratio" used to represent "inelastic action" would have to reach values not consonant with present information. Some of the ratios indicated by comparing values of V with spectra indicate that the framers of the SEAOC proposal anticipated damping approaching the "over-damped" or "dead-beat" condition.

This has a direct bearing upon another of the author's parallels—the comparison of F_x, the "distributed base shear" of the SEAOC proposal, with F_{xn},

10 "Behavior of Structures During Earthquakes," by G. W. Housner, Proceedings, ASCE, Vol. 85, No. EM 4, October, 1959, p. 128.

the dynamic counterpart. These are shown as functions of V and V_n, respectively. Hence, their ratio, $\frac{F_{xn}}{F_x} = \frac{V_n}{V}$.

Two practical problems are involved. One of these is the use of F_x as a criterion of probable story distortion, during "moderate" earthquakes, to control "non-structural" damage. It is evident that, if the ratio, $\frac{V_n}{V}$, is greater than 2.0, the computed distortion will be less than the probable maximum and "non-structural" damage may result. The ratios deduced from the SEAOC proposal indicate that extensive "non-structural" damage should be developed by a "moderate" earthquake and, that, following "extremely strong ground motion," practically nothing but a frame-work, probably severely distorted, will survive.

The other problem is more serious, in one type of building.

In the SEAOC proposal, the reduced values F_x are used to compute M, the "overturning" moment. The true value of M is the sum of the moments of the actual "equivalent inertia forces" capable of producing the maximum dynamic distortion, as maximum response. The resultants of both F_x and F_{xn} act at approximately 2/3 of the height of the structure and, from the ratio, $\frac{V_n}{V}$, the $\sum F_x h_x$ is $\frac{V_{xn}}{V}\left(\sum F_{xn} h_x\right)$. Hence, to evaluate the maximum "overturning" moment from F_x, there should be a correction factor, increasing the $\sum F_x h_x$ by $\frac{V_n}{V}$. The SEAOC proposal provides a correction factor, but it decreases the summation by $J = \frac{0.5}{\sqrt[3]{T^2}}$ with limits 1.0 to 0.33.

The factor J, in effect, lowers the position of the resultant of the reduced lateral loading, $\sum F_x$, from the position established by the assumption of "triangular distribution," for example, approximately 2/3 of the height, to as low as 2/9 H. No adequate explanation has yet been offered for this procedure.

It is the actual "equivalent inertia force" that must be considered in designing at "near-collapse stress" levels. At maximum distortion, both the normal dead and live load and the axial "overturning" load are applied eccentrically to the wall columns. In the critical first story, this eccentricity is a function of V_n, the actual maximum base shear, and not of V, the reduced quantity reflecting design for "moderate" ground motion. Moreover, this eccentric loading is applied to columns under substantial bending strain produced by the actual maximum base shear, V_n.

This combination of strain is particularly critical in tall, narrow-based buildings. It is not unlikely that, in such structures, complying with the SEAOC proposal, the response action of first story columns might terminate, abruptly and decisively, before any considerable relief could be developed by "inelastic action" in upper stores.

The aim of the Seismology Committee appears to have been production of a set of simple rules and formulas, equally applicable in all situations, and to all materials and methods of construction. In its present development, the art of aseismic design does not appear to have reached the point of such accomplishment. The author's justification of the SEAOC proposal, on the ground of consonance with dynamic theory, does not seem to warrant the strong, general approval which he expresses.

GEORGE W. HOUSNER.[11]—The author has given a very clear explanation of the dynamic effects of earthquake ground motion and of the manner in which the proposed SEAOC code provisions are related to the dynamic analysis of structures. In view of the fact that the SEAOC provisions will likely supersede those now in use, the writer would like to place on record some historical remarks about the development of earthquake codes, particularly with respect to the original Los Angeles (LA) code provisions that were adopted in 1940.

The Los Angeles provisions applied only to structures not exceeding 150 ft in height. The base shear of a structure was specified to be

$$V = \frac{0.6}{n + 3.5} W \quad \text{. . . . . (21)}$$

in which n is the number of stories of the structure and W is the total weight of the structure. This expression for V can be compared with that of the SEAOC code if use is made of the empirical formula for the fundamental period

TABLE 2.—COMPARISON OF BASE SHEARS OF LOS ANGELES AND SEAOC CODES

Los Angeles	SEAOC
$V = \frac{0.06}{T + 0.35} W$	$V = \frac{0.05}{(T)^{1/3}} W$
$V = \frac{2\pi}{T} \frac{W}{g} S_v$	$V = \frac{2\pi}{T} \frac{W}{g} S_v$
$S_v = 0.31 \frac{T}{T + 0.35}$	$S_v = 0.26 (T)^{2/3}$

of vibration, $T = 0.1 n$, to express n in terms of T. Table 2 compares the expressions for V and corresponding expressions for the velocity spectrum S_v, with $K = 1$. The two curves of S_v are shown in the accompanying diagram where they are drawn on a graph of the average velocity spectra.[12] The LA curve is coincident with the 40% of critical damping curve. It is seen that the SEAOC and the LA curves are similar with the SEAOC giving smaller values below $T = 0.9$ sec.

It is seen that the LA code incorporates the spectrum concept but with the period not appearing explicitly. It was explained to the writer by R. R. Martel that it was felt, at the time the LA code was formulated, that expressions involving T explicitly might tend to mislead the designer into overestimating the accuracy and logic of his computations.

The distribution of shear over the height of a building was specified in the LA code to be

$$V_i = \frac{0.6}{N + 4.5} W_i \; (N \gtrsim 12) \quad \text{. . . . . (22)}$$

[11] Prof. of Engrg., Calif. Inst. of Tech., Pasadena, Calif.

[12] "Behavior of Structures During Earthquakes," by G. W. Housner, Proceedings, ASCE, Vol. 85, EM 4, October, 1959.

in which V_i is the shear in the i-th story from the top, W_i is the total weight above the midheight of the i-th story, and N is the number of stories above the story under consideration. The SEAOC code states that $V_i = \sum F_x$ where $F_x = V \frac{W_x h_x}{\sum wh}$ as explained by the author. The LA code was limited to buildings of thirteen stores or less. A comparison of the provisions of the SEAOC and LA codes, indicating the distribution of shears for a uniform ten-story building, with each floor and roof of equal mass, is shown in Fig. 10, with the base shear the same in both cases. It is seen that the distributions are very similar with the SEAOC curve giving somewhat smaller shears in the upper parts of the structure.

It is, thus, seen that over the range of applicability of the LA code, the provisions of the SEAOC code are essentially the same, the chief difference being that the SEAOC code gives somewhat smaller values for base shears and for the shears over the height. The SEAOC code extends the provisions to structures exceeding 150 ft in height and expresses the provisions in different language.

A new feature of the code is the prescription of different seismic factors for buildings of different types. This is an attempt to take into account the different capabilities of structures to absorb vibrational energy by plastic deformation. The writer would like to emphasize the author's statement concerning the desirability of additional study and research on this point. As illustration of this need, it can be pointed out that for a building that can absorb energy plastically, and for ground motion whose velocity spectrum is essentially constant, there is evidence that the seismic design forces should be independent of the period of vibration, if the safety factor is based on collapse. On the other hand, if the safety factor is based on some maximum stress or strain, the seismic forces should be dependent upon the period of vibration.

Since the velocity response spectrum[12,13] is now playing a large role in earthquake and explosive-induced ground shock problems, it is perhaps in order to set down for the record a brief history of its development. The first case of an attempt to determine a spectrum was in 1857, when Robert Mallet constructed a falling-pin seismometer.[14] This instrument consisted of a set of cylindrical rods of different lengths standing on end. When the earthquake strikes, the pins oscillate and some of them may fall over, thus giving an indication of the character of the ground motion. Since the oscillating pin is a non-linear, inverted pendulum, the falling-pin seismometer is actually a type of spectrum instrument which, however, suffers from the difficulty of assigning a meaning to the results.

After the great Tokyo earthquake in 1923, K. Suyehiro[15] constructed an instrument consisting of thirteen ordinary, lightly damped pendulums of different periods whose maximum amplitudes of vibration during an earthquake were recorded. This instrument was thus the forerunner of the reed-gage, and the thirteen data points given by it determine the displacement-response-spectrum. A similar instrument was later constructed by the United States Coast and Geodetic Survey, (USC&GS) Dept. of Commerce in San Francisco,

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13 "The Response Spectrum Technique," by D. E. Hudson, Proceedings of the 1956 World Conf. on Earthquake Engrg., Earthquake Engrg. Research Inst., San Francisco, Calif., 1956.

14 "The Great Neapolitan Earthquake of 1857," by R. Mallet, London, 1862.

15 "Engineering Seismology," by K. Suyehiro, Proceedings, ASCE, Vol. 58, No. 4, May, 1932.

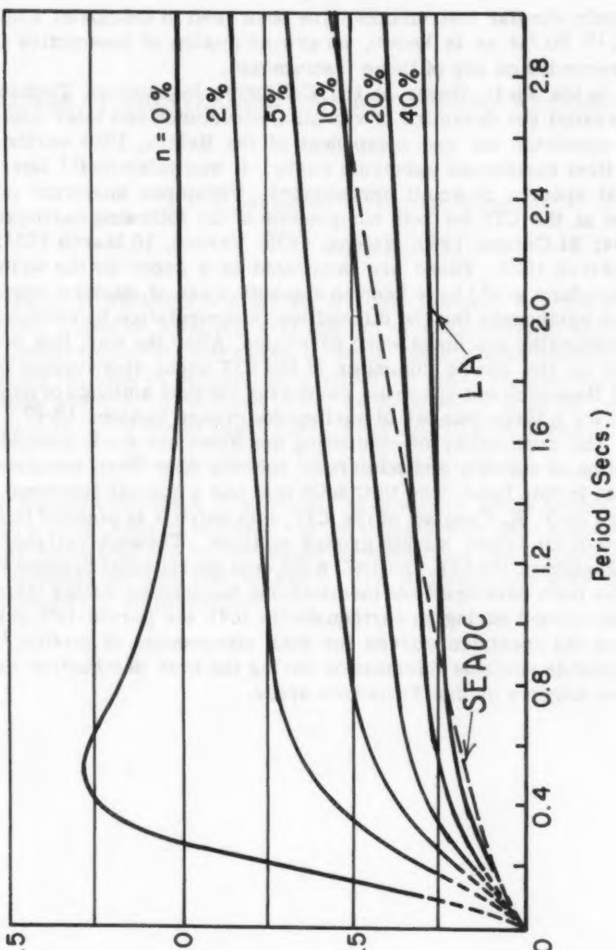

FIG. 11.—AVERAGE VELOCITY SPECTRA

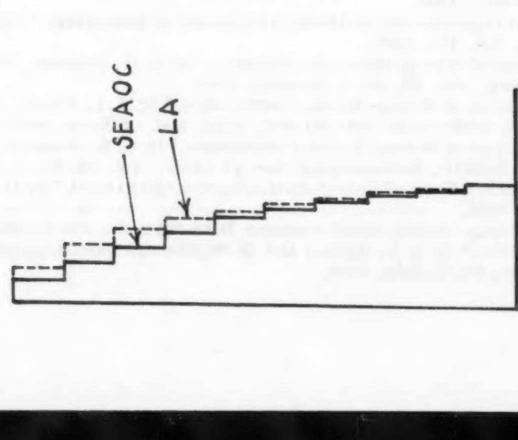

FIG. 10.—SEISMIC SHEARS TEN STORY BUILDING

and more recently similar instruments have been used in the Soviet Union by A. G. Nazarov.[16] So far as is known, no ground motion of destructive intensity has been recorded on any of these instruments.

M. A. Biot, in his Ph.D. thesis at the California Institute of Technology, (CIT) (1932) treated the dynamics of vibrating structures and later computed the undamped spectrum for one component of the Helena, 1935 earthquake. This was the first earthquake spectrum curve. It was published[17] later with some additional spectra of small earthquakes. Undamped spectrum curves were computed at the CIT for both components of the following earthquakes: El Centro, 1934; El Centro, 1940; Helena, 1935; Vernon, 10 March 1933; Sub Terminal, 10 March 1933. These are described in a paper by the writer.[18] The logical procedure would have been to compute a set of damped spectrum curves for each earthquake but the difficulties in computation forestalled this until suitable computing machines were developed. After the war, this project was undertaken on the analog computer at the CIT under sponsorship by the Office of Naval Research and spectrum curves for various amounts of damping were computed for a large number of earthquake ground motions.[19,20]

Since 1950, the availability of computing machines has made possible the ready computation of spectra and numerous spectra have been computed by various workers in this field. The USC & GS now has a special spectrum analyzer, developed by T. K. Caughey of the CIT, with which it is planned to compute the spectra of all future strong ground motions. Through collaboration between D. E. Hudson at the CIT, the USC & GS, and the National Science Foundation there have been developed and installed one hundred so-called 'seismoscopes' that will record during an earthquake the 0.75 sec period-10% critical damping point on the spectrum curves for both components of motion.[21,22] These should provide valuable information during the next destructive earthquake in the Los Angeles or San Francisco areas.

16 "Analytical Methods of Engineering Seismology," by A. G. Nazarov, Academy of Science of Armenian SSR, 1959.

17 "Analytical and Experimental Methods in Engineering Seismology," by M. A. Biot, Transactions, ASCE, Vol. 108, 1943.

18 "Characteristics of Strong-Motion Earthquakes," by G. W. Housner, Bulletin Seismological Soc. of Amer., Vol. 37, No. 1, January, 1947.

19 "Spectrum Analysis of Strong-Motion Earthquakes," by J. L. Alford, G. W. Housner, and R. R. Martel, ONR Report NR-081-095, Calif. Inst. of Tech., 1951.

20 "Spectrum Analysis of Strong-Motion Earthquakes," by G. W. Housner, R. R. Martel, and J. L. Alford, Bulletin, Seismological Soc. of Amer., Vol. 43, No. 2, April, 1953.

21 "The Wilmot-Survey Type of Strong-Motion Earthquake Record," by D. E. Hudson, Calif. Inst. of Tech., 1958.

22 "Analysis of Strong-Motion Accelerometer Data from the San Francisco Earthquake of March 22, 1957," by D. E. Hudson and G. W. Housner, Bulletin, Seismological Soc. of Amer., Vol. 48, No. 2, July, 1958.

DESIGN OF PRESTRESSED CONCRETE BEAMS BY COMPUTER[a]

Discussion by Richard J. Newson

RICHARD J. NEWSON,[2] A.M. ASCE.—Mr. Bonasia has presented a very interesting paper on a subject that has current significance in that it is fairly easy to understand. If called upon to do so, the engineer should have no trouble solving the kind of problem with which the paper is concerned.

A small error is to be found in Eq. 26 because, if h is in inches, as apparently it is, 26/8 would equal 3.25. Thus Eq. 26 should read

$$V = 0.08(4L - h) + 3.24(8L - h)/L \qquad (26)$$

Mr. Bonasia assumes that proper keys and/or shear connectors are provided to prevent failure in shear. It is thought that Mr. Bonasia could have used the Bendix G-15D to advantage by programming a check on shear flow, V Q/I, between the slab and the stem. The machine will give the spacing of the connectors if so programed.

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[a] April, 1960, by Joseph J. Bonasia.

[2] Design Engr., Indiana State Highway.

[illegible]

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CHARTS FOR DESIGN OF REINFORCED CONCRETE COLUMNS[a]

Discussion by George B. Begg, Jr.

GEORGE B. BEGG, JR.,[12] M. ASCE.—The authors have presented valuable charts for the design of reinforced concrete columns. These design aids certainly will help their avowed purpose of "breaking down resistance to the use of ultimate strength design."

However, the writer questions whether the units p_t and e' are variables that should be plotted. As the writer pointed out previously,[13] the introduction of such variables only represent intermediate steps toward a solution that can me made more directly. For example, at the end of a normal analysis of a floor system, the ultimate load and moment on a column are known. The design question becomes "What column size and reinforcement are required by this load and moment? This being the question, why not replace e' with M_u and plot actual bar arrangements instead of values of p_t?

It the charts are extended (and it is to be hoped that they will be) to different bar arrangements, column shapes and higher stresses, a set of indexes might help a more rapid convergence. A typical index would plot the upper line (pt or given bars) for each column size of a constant concrete and steel stress. The designer, knowing his design stresses, goes to the index and picks out the smallest column for his load and moment. In turn, he moves to the applicable chart and picks out the reinforcement.

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[a] May, 1960, by W. H. Gardener, Jr., and Donald H. Kline.

[12] Chf., Design and Research Sect., Structural Engrg. Branch, Publ. Bldgs. Service, Genl. Services Admin.

[13] Discussion of "Design of Symmetrical Columns with Small Eccentricities in One or Two Directions," Journal, ACI, Vol. 30, No. 9, March, 1959.

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ERRATA

Journal of the Structural Division

Proceedings of the American Society of Civil Engineers

May, 1960

p. 12. In the line above Table 1 change p_t = 0.021 to p_t = 0.92).

p. 12. In Table 1 delete the value under an 18 in. by 18 in. column that corresponds to Ten #11 bars.

p. 12. In the second line under Table 1 change the word "inadequate" to "adequate."

June, 1960

p. 79. Immediately after the heading "Material Properties: Bolts" change "The bolts used. . ." to "The ASTM-A325 bolts used. . ."

p. 90. Delete the last sentence in the second full paragraph and substitute therefore "The bolt failure of joint A3 is shown in Fig. 10."

p. 93. From the two lines just above Fig. 13, delete the words "This is so because" so that the sentence begins "The displacement is due. . ."

p. 93. From the eighth line below Fig. 13, delete the words "that is" and substitute therefore a comma.

p. 93. In the ninth line below Fig. 13, change "which" to "while."

p. 96. Just prior to the sixth line of text from the bottom of the page, insert the centered heading SUMMARY AND CONCLUSIONS.

PROCEEDINGS PAPERS

The technical papers published in the past year are identified by number below. Technical-division sponsorship is indicated by an abbreviation at the end of each Paper Number, the symbols referring to: Air Transport (AT), City Planning (CP), Construction (CO), Engineering Mechanics (EM), Highway (HW), Hydraulics (HY), Irrigation and Drainage (IR), Pipeline (PL), Power (PO), Sanitary Engineering (SA), Soil Mechanics and Foundations (SM), Structural (ST), Surveying and Mapping (SU), and Waterways and Harbors (WW), divisions. Papers sponsored by the Department of Conditions of Practice are identified by the symbols (PP). For titles and order coupons, refer to the appropriate issue of "Civil Engineering." Beginning with Volume 82 (January 1956) papers were published in Journals of the various Technical Divisions. To locate papers in the Journals, the symbols after the paper number are followed by a numeral designating the issue of a particular Journal in which the paper appeared. For example, Paper 2270 is identified as 2270(ST9) which indicates that the paper is contained in the ninth issue of the Journal of the Structural Division during 1959.

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AUGUST: 2126(HY8), 2127(HY8), 2128(HY8), 2129(HY8), 2130(PO4), 2131(PO4), 2132(PO4), 2133(PO4), 2134 (SM4), 2135(SM4), 2136(SM4), 2137(SM4), 2138(HY8)c, 2139(PO4)c 2140(SM4)c.

SEPTEMBER: 2141(CO2), 2142(CO2), 2143(CO2), 2144(HW3), 2145(HW3), 2146(HW3), 2147(HY9), 2148(HY9), 2149(HY9), 2150(HY9), 2151(IR3), 2152(ST7)c, 2153(IR3), 2154(IR3), 2155(IR3), 2156(IR3), 2157(IR3), 2158 (IR3), 2159(IR3), 2160(IR3), 2161(SA5), 2162(SA5), 2163(ST7), 2164(ST7), 2165(SU1), 2166(SU1), 2167(WW3), 2168(WW3), 2169(WW3), 2170(WW3), 2171(WW3), 2172(WW3), 2173(WW3), 2174(WW3), 2175(WW3), 2176 (WW3), 2177(WW3), 2178(CO2)c, 2179(IR3)c, 2180(HW3)c, 2181(SA5)c, 2182(HY9)c, 2183(SU1)c, 2184 (WW3)c, 2185(PP2)c, 2186(ST7)c, 2187(PP2), 2188(PP2).

OCTOBER: 2189(AT4), 2190(AT4), 2191(AT4), 2192(AT4), 2193(AT4), 2194(EM4), 2195(EM4), 2196(EM4), 2197(EM4), 2198(EM4), 2199(EM4), 2200(HY10), 2201(HY10), 2202(HY10), 2203(PL3), 2204(PL3), 2205 (PL3), 2206(PO5), 2207(PO5), 2208(PO5), 2209(PO5), 2210(SM5), 2211(SM5), 2212(SM5), 2213(SM5), 2214 (SM5), 2215(SM5), 2216(SM5), 2217(SM5), 2218(ST8), 2219(ST8), 2220(EM4), 2221(ST8), 2222(ST8), 2223 (ST8), 2224(HY10), 2225(HY10), 2226(PO5), 2227(PO5), 2228(PO5), 2229(ST8), 2230(EM4), 2231(EM4), 2232(AT4)c, 2233(PL3)c, 2234(EM4)c, 2235(HY10)c, 2236(SM5)c, 2237(ST8)c, 2238(PO5)c, 2239(ST8), 2240 (PL3).

NOVEMBER: 2241(HY11), 2242(HY11), 2243(HY11), 2244(HY11), 2245(HY11), 2246(SA6), 2247(SA6), 2248 (SA6), 2249(SA6), 2250(SA6), 2251(SA6), 2252(SA6), 2253(SA6), 2254(SA6), 2255(SA6), 2256(ST9), 2257(ST9), 2258(ST9), 2259(ST9), 2260(HY11), 2261(ST9)c, 2262(ST9), 2263(HY11), 2264(ST9), 2265(HY11), 2266(SA6), 2267(SA6), 2268(SA6), 2269(HY11)c, 2270(ST9).

DECEMBER: 2271(HY12)c, 2272(CP2), 2273(HW4), 2274(HW4), 2275(HW4), 2276(HW4), 2277(HW4), 2278 (HW4), 2279(HW4), 2280(HW4), 2281(IR4), 2282(IR4), 2283(IR4), 2284(IR4), 2285(PO6), 2286(PO6), 2287 (PO6), 2288(PO6), 2289(PO6), 2290(PO6), 2291(PO6), 2292(SM6), 2293(SM6), 2294(SM6), 2295(SM6), 2296 (SM6), 2297(WW4), 2298(WW4), 2299(WW4), 2300(WW4), 2301(WW4), 2302(WW4), 2303(WW4), 2304(HW4), 2305(ST10), 2306(CP2), 2307(CP2), 2308(ST10), 2309(CP2), 2310(HY12), 2311(HY12), 2312(PO6), 2313(PO6), 2314(ST10), 2315(HY12), 2316(HY12), 2317(HY12), 2318(WW4), 2319(SM6), 2320(SM6), 2321(ST10), 2322 (ST10), 2323(HW4)c, 2324(CP2)c, 2325(SM6)c, 2326(WW4)c, 2327(IR4)c, 2328(PO6)c, 2329(ST10)c, 2330 (CP2).

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JANUARY: 2331(EM1), 2332(EM1), 2333(EM1), 2334(EM1), 2335(HY1), 2336(HY1), 2337(EM1), 2338(EM1), 2339(HY1), 2340(HY1), 2341(SA1), 2342(EM1), 2343(SA1), 2344(ST1), 2345(ST1), 2346(ST1), 2347(ST1), 2348(EM1)c, 2349(HY1)c, 2350(ST1), 2351(ST1), 2352(SA1)c, 2353(ST1)c, 2354(ST1).

FEBRUARY: 2355(CO1), 2356(CO1), 2357(CO1), 2358(CO1), 2359(CO1), 2360(CO1), 2361(PO1), 2362(HY2), 2363(ST2), 2364(HY2), 2365(SU1), 2366(HY2), 2367(SU1), 2368(SM1), 2369(HY2), 2370(SU1), 2371(HY2), 2372(PO1), 2373(SM1), 2374(HY2), 2375(PO1), 2376(HY2), 2377(CO1)c, 2378(SU1), 2379(SU1), 2380(SU1), 2381(HY2)c, 2382(ST2), 2383(SU1), 2384(ST2), 2385(SU1)c, 2386(SU1), 2387(SU1), 2388(SU1), 2389(SM1), 2390(ST2)c, 2391(SM1)c, 2392(PO1)c.

MARCH: 2393(IR1), 2394(IR1), 2395(IR1), 2396(IR1), 2397(IR1), 2398(IR1), 2399(IR1), 2400(IR1), 2401(IR1), 2402(IR1), 2403(IR1), 2404(IR1), 2405(IR1), 2406(IR1), 2407(SA2), 2408(SA2), 2409(HY3), 2410(ST3), 2411 (SA2), 2412(HW1), 2413(WW1), 2414(WW1), 2415(HY3), 2416(HW1), 2417(HW3), 2418(HW1)c, 2419(WW1)c, 2420(WW1), 2421(WW1), 2422(WW1), 2423(WW1), 2424(SA2), 2425(SA2)c, 2426(HY3)c, 2427(ST3)c.

APRIL: 2428(ST4), 2429(HY4), 2430(PO2), 2431(SM2), 2432(PO2), 2433(ST4), 2434(EM2), 2435(PO2), 2436 (ST4), 2437(ST4), 2438(HY4), 2439(EM2), 2440(EM2), 2441(ST4), 2442(SM2), 2443(HY4), 2444(ST4), 2445 (EM2), 2446(ST4), 2447(EM2), 2448(SM2), 2449(HY4), 2450(ST4), 2451(HY4), 2452(HY4), 2453(EM2), 2454 (EM2), 2455(EM2)c, 2456(HY4)c, 2457(PO2)c, 2458(ST4)c, 2459(SM2)c.

MAY: 2460(AT1), 2461(ST5), 2462(AT1), 2463(AT1), 2464(CP1), 2465(CP1), 2466(AT1), 2467(AT1), 2468(SA3), 2469(HY5), 2470(ST5), 2471(SA3), 2472(SA3), 2473(ST5), 2474(SA3), 2475(ST5), 2476(SA3), 2477(ST5), 2478 (HY5), 2479(SA3), 2480(ST5), 2481(SA3), 2482(CO2), 2483(CO2), 2484(HY5), 2485(HY5), 2486(AT1)c, 2487 (CP1)c, 2488(CO2)c, 2489(HY5)c, 2490(SA3)c, 2491(ST5)c, 2492(CP1), 2493(CO2).

JUNE: 2494(IR2), 2495(IR2), 2496(ST6), 2497(EM3), 2498(EM3), 2499(EM3), 2500(EM3), 2501(SM3), 2502 (EM3), 2503(PO3), 2504(WW2), 2505(EM3), 2506(HY6), 2507(WW2), 2508(PO3), 2509(ST6), 2510(EM3), 2511 (EM3), 2512(ST6), 2513(HW2), 2514(HY6), 2515(PO3), 2516(EM3), 2517(WW2), 2518(WW2), 2519(EM3), 2520 (PO3), 2521(HY6), 2522(SM3), 2523(ST6), 2524(HY6), 2525(HY6), 2526(HY6), 2527(IR2), 2528(ST6), 2529 (HW2), 2530(IR2), 2531(HY6), 2532(EM3)c, 2533(HW2)c, 2534(WW2), 2535(HY6)c, 2536(IR2)c, 2537(PO3)c, 2538(SM3)c, 2539(ST6)c, 2540(WW2)c.

JULY: 2541(ST7), 2542(ST7), 2543(SA4), 2544(ST7), 2545(ST7), 2546(HY7), 2547(ST7), 2548(SU2), 2549(SA4), 2550(SU2), 2551(HY7), 2552(ST7), 2553(SU2), 2554(SA4), 2555(ST7), 2556(SA4), 2557(SA4), 2558(SA4), 2559(ST7), 2560(SU2)c, 2561(SA4)c, 2562(HY7)c, 2563(ST7)c.

AUGUST: 2564(SM4), 2565(EM4), 2566(ST8), 2567(EM4), 2568(PO4), 2569(PO4), 2570(HY8), 2571(EM4), 2572(EM4), 2573(EM4), 2574(SM4), 2575(EM4), 2576(EM4), 2577(HY8), 2578(EM4), 2579(PO4), 2580 (EM4), 2581(ST8), 2582(ST8), 2583(EM4)c, 2584(PO4)c, 2585(ST8)c, 2586(SM4)c, 2587(HY8)c.

c. Discussion of several papers, grouped by divisions.

ST8

PART 2

AUGUST 1960 — 26
VOLUME 86

NO. ST 8
PART 2

Your attention is invited

NEWS OF THE STRUCTURAL DIVISION OF ASCE

AMERICAN SOCIETY OF CIVIL ENGINEERS FOUNDED 1852

JOURNAL OF THE STRUCTURAL DIVISION
PROCEEDINGS OF THE AMERICAN SOCIETY OF CIVIL ENGINEERS

DIVISION ACTIVITIES

STRUCTURAL DIVISION

Proceedings of the American Society of Civil Engineers

Contents

NEWS

August, 1960

BOSTON ASCE CONVENTION

The Annual Convention of the ASCE will be held in Boston, Mass., from October 9 to 14, 1960. The theme of the Structural Division Program will be Basic Research. Saul Namyet, Structural Division Local Chairman on Sessions Programs, reports that the program is essentially complete. The complete program of the Convention will be published in CIVIL ENGINEERING.

A special innovation is being introduced for the first time at this meeting. Two sessions of the Structural Division Program will be devoted to short informal reports on progress in current research efforts. For a period of two days, approximately 25 different research programs will be described briefly. Considerable time will be allowed for discussion and it is hoped that people working in related areas will be prepared to join the discussion.

One of the highlights of the program will be a paper by Dr. Gene M. Nordby entitled "Men, Money, Research and the University." It will cover the general aspects of government research, aspects of proposal writing and the government's philosophy in supporting basic research in engineering.

In addition to Dr. Nordby's paper and the short informal reports on current research, the following seven papers will be presented:

Analysis of Structural Safety
    Ernest Basler, Switzerland

Note.—No. 1960-26 is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 86, No. ST 8, August, 1960.

Safety, Reliability, and Structural Design
Dr. A. M. Freudenthal, Columbia University

Applied Statistics in Experimental Investigation
Dr. Paul Irick, AASHO Road Test

Post-Elastic Flexural Behavior of Rolled Steel Sections
Herbert A. Sawyer, Jr., University of Connecticut
Yechiel Weitsman, University of Connecticut

Dynamic Behavior of Girder Bridges
Edward N. Wilson, New York University
James Michalos, New York University

Wind Tunnel Tests on Wind-Induced Vibrations in Structural Members
Dr. William Weaver, Jr., Stanford University

The Application of Finite Difference to Some Structural Problems
Frederick G. Lehman, Newark College of Engineering

AWS MEETING

The national fall meeting of the American Welding Society will be held at the Hotel Penn-Sheraton, Pittsburgh, Pennsylvania, during the week of September 26-29. There are 17 half-day technical sessions planned for this meeting. Of special interest to structural engineers are the following papers on welded structures:

September 26, Monday Afternoon

A. Welding of High-strength Low-alloy Structural Steels for Bridges
by John L. Beaton and Paul G. Jonas, California Division of Highways
B. Behavior of Welded Built-up Beams Under Repeated Loads
by J. E. Stallmeyer, L. R. Hall and W. H. Munse, University of Illinois
C. The Flexural Fatigue Strength of Stiffened Beams
by J. E. Stallmeyer and W. H. Munse, University of Illinois

September 27, Tuesday Morning

A. Tests of Welded Plate Girders
by B. T. Yen, Lehigh University
B. Design of Columns in Multi-story Frames
by J. S. Ellis, Royal Military College of Canada
C. Welded Interior Beam-to-Column Connections
by J. D. Graham, A. N. Sherbourne, R. N. Khabbaz, and C. D. Jensen, Lehigh University

September 27, Tuesday Afternoon

A. Design Features for Welded Joints in Space Trusses for Retractable-roof Auditorium
by Edward Cohen, Ammann & Whitney
B. Plastic Design of Fixed-base, Gabled Frames
by S. P. Prawel, Jr., and R. L. Ketter, University of Buffalo
C. Behavior of Haunched Connections
by G. C. Lee, J. W. Fisher and G. C. Driscoll, Jr., Lehigh University

September 28, Wednesday Afternoon

A. Strength of Welded Aluminum -alloy Box Beams
by R. J. Brungraber, Alcoa Research Laboratories

B. Research and Development of Continuous Welded Rail
by G. M. Magee, Association of American Railroads

C. Painting of Welds
by J. D. Keane, Steel Structures Painting Council

The Monday and Wednesday afternoon sessions are sponsored by the Structural Division of the ASCE, and the Tuesday sessions are co-sponsored by the Column Research Council and the Structural Division of the ASCE. The complete and detailed program of the meeting will be published in the September issue of the WELDING JOURNAL.

NEW COMMITTEE POLICY IN PRESTRESSED CONCRETE FIELD

After the 1958 report was developed by the Joint ASCE-ACI Committee on Prestressed Reinforced Concrete under the Chairmanship of Mr. Thor Germundsson, it was recognized that the existing committee structure with over forty individual members had become unwieldy. At the recommendation of that committee plans were made for a change in committee organization. The old committee was discharged earlier this year with thanks for a very difficult job which they had successfully completed.

The organization of a new committee has now been completed and has been approved by both ASCE and ACI. This new committee under the Chairmanship of Mr. Morris Schupack of the firm of Schupack and Zollman, Stamford, Connecticut, will be called the Joint ACI-ASCE Committee on Prestressed Concrete. The mission of this committee is officially stated as follows:

> To continuously evaluate the technical status of prestressed concrete and to recommend to the joint sponsors the formation of any needed technical committees or subcommittees for the study of particular developments, innovations, or problems arising in this field. Such committee activity should be directed toward the publishing of information and the development of tentative recommendations or recommended practices which incorporate new procedures or correct existing deficiencies. The committee will also study significant research problems and make recommendations.

The new committee will be a policy committee and both societies will lean heavily upon its recommendations with regard to the creation of additional task committees in this area. It is expected that a number of these task committees, each working on a special problem in the field of prestressed concrete, will eventually result. In ASCE this committee now acts as a task committee within the Committee on Masonry and Reinforced Concrete under the Chairmanship of Professor Phil M. Ferguson.

The complete committee is as follows:

Morris Schupack,* Schupack and Zollman, Stamford, Connecticut, Chairman

W. B. Bennett, Structural Bureau, Portland Cement Association, Chicago, Illinois

John N. Clary, Bridge Engineer, Virginia Department of Highways, Richmond, Virginia

Ben C. Gerwick,* Ben C. Gerwick, Inc., San Francisco, California

Jack Janney, The Engineers Collaborative, Des Plaines, Illinois

T. Y. Lin, University of California, Berkeley, California

Alan H. Mattock, Senior Research Engineer, Portland Cement Association, Skokie, Illinois

Chester P. Siess,* University of Illinois, Urbana, Illinois

Wendell R. Wilson,* Assistant Engineer, Atchison, Topeka and Santa Fe Railroad, Chicago, Illinois

The control group, starred in the above list, is planning an organization meeting in Chicago in early September.

CONFERENCE ON ELECTRONIC COMPUTATION

The Second National Conference on Electronic Computation is scheduled for September 7-9, 1960 at the Pittsburgh-Hilton Hotel. The Conference is sponsored by the Structural Division and the Pittsburgh Section of the ASCE.

One of the features of the Conference will be a computer exhibit which will include certain items of new equipment having their first public exhibition. Among the companies participating are Friden, International Business Machines, National Cash Register and Bendix. On Wednesday and Thursday evenings, these same manufacturers will present computer seminars, for the benefit of the Conference participants.

Jackson L. Durkee, Chairman of the Program Committee and those who assisted him are to be congratulated for preparing a very fine program. The three concurrent sessions on Thursday and two on Friday have been arranged to allow Conference participants to select the items of greatest interest. The technical sessions are as follows:

THURSDAY MORNING 8 SEPTEMBER 1960

One general session followed by three simultaneous sessions

Session A1—Ballroom 2

Presiding: E. J. Ruble, Vice Chairman, Executive Committee, Structural Division

10:30 A.M. Computer Program Exchange: Myth and Reality
Jerry C. L. Chang, Chief Design Engineer, Richardson, Gordon & Associates, Pittsburgh, Penna.

11:10 A.M. Computer Design of Structural Steel for Buildings
M. Zar, Associate, and C. Beck, Design Engineer, Sargent & Lundy, Engineers, Chicago, Ill.

Session A2—Ballroom 3

Presiding: James Michalos, Chairman, Task Committee on Program Exchange, Committee on Electronic Computation, Structural Division

10:30 A.M. Computer Solutions to Linear Buckling Problems
Richard J. Sylvester, Assistant Research Scientist, The Martin Co., Denver, Colorado

11:10 A.M. Error Analysis for Eigenvalue Problems
Robert B. McCalley, Jr., Consulting Engineer, Knolls Atomic Power Laboratory, General Electric Company, Schenectady, N. Y.

Session A3—Ballroom 4
Presiding: S. J. Fenves, Secretary, Committee on Electronic Computation, Structural Division

10:30 A.M. Optimum Design of Transmission Towers
George P. Anaston, Blaw-Knox Equipment Division, Blaw-Knox Co., Pittsburgh, Penna.

11:10 A.M. Optimum Design of Reinforced Concrete Buildings
John D. Graham, Consulting Civil Engineer, KCS Limited, Toronto, Ontario

THURSDAY AFTERNOON 8 SEPTEMBER 1960
Three simultaneous sessions

Session B1—Ballroom 2
Presiding: Jackson L. Durkee, Chairman, Task Committee on Conferences, Committee on Electronic Computation, Structural Division

2:20 P.M. Computer Analysis of Structures
J. Ferry Borges, Head, Structural Studies Section, Laboratorio Nacional de Engenharia Civil, Lisbon, Portugal

3:00 P.M. Structural Idealization for Digital Computer Analysis
John S. Archer and Charles H. Samson, Jr., Project Structures Engineers, Convair, Fort Worth, Texas

3:40 P.M. Structural Design by Systematic Synthesis
Lucien A. Schmit, Jr., Ass't. Professor of Structures, Case Institute of Technology, Cleveland, Ohio

4:20 P.M. Simultaneous Equations Solved by Over-Relaxation
Frederick G. Lehman, Assoc. Professor in Civil Engineering, Newark College of Engineering, Newark, N. J.

Session B2—Ballroom 3
Presiding: R. A. Reickert, Vice Chairman, Task Committee on Program Directory and Library, Committee on Electronic Computation, Structural Division

2:20 P.M. The Electronic Computer as a Tool in Moment Distribution
J. E. Soehrens, Staff Consultant, C. F. Braun & Co., Alhambra, Calif.

3:00 P.M. Multi-Story Frame Analysis by Digital Computer
Moshe F. Rubinstein, Engineer, Victor Gruen Associates, Los Angeles, California

3:40 P.M. Stiffness Method of Rigid Frame Analysis
Ming L. Pei, Assoc. Professor of Civil Engineering, The City College, New York, N. Y.

4:20 P.M. Computing Maximums Due to Moving Loads
James N. Lingeman, Engineer, Hazelet & Erdal, Chicago, Ill.

Session B3—Ballroom 4

Presiding: Glen V. Berg, Chairman, Task Committee on Mathematical Methods, Committee on Electronic Computation, Structural Division

2:20 P.M. Some Fundamental Concepts and Their Use in Matrix Structural Analysis
Frank R. Berman, Consulting Engineer, Huntington, N. Y.

3:00 P.M. The Elements of Matrix Structural Analysis
Sidney Shore, Professor of Civil Engineering, University of Pennsylvania, Philadelphia, Pennsylvania

3:40 P.M. Dynamic Analysis of Circular Arches
R. T. Eppink, Research Associate, and A. S. Veletsos, Professor of Civil Engineering, University of Illinois, Urbana, Illinois

4:20 P.M. A Numerical Procedure for the Analysis of Continuous Plates
Alfredo H. S. Ang, Ass't. Professor of Civil Engineering, and Nathan M. Newmark, Head, Dept. of Civil Engineering, University of Illinois, Urbana, Ill.

FRIDAY MORNING 9 SEPTEMBER 1960

Two simultaneous sessions

Session C1—Ballroom 2

Presiding: Elmer K. Timby, Director, ASCE

9:00 A.M. Sloping Surcharge Retaining Wall Design
Maurice A. Wadsworth, Computing Engineer, Gannett Fleming Corddry & Carpenter, Harrisburg, Penna.

9:40 A.M. Vessel Foundation Designs Using a Digital Computer
L. G. Horton, Head of Civil Engineering Dept., and E. S. Eichmann, Engineering Analyst, C. F. Braun & Co., Alhambra, Calif.

10:20 A.M. Stress Distribution Patterns and Settlement Characteristics of Structural Pile Foundations
Eugene P. Rausa, Soils Engineer, Howard, Needles, Tammen & Bergendoff, New York, N. Y.

11:00 A.M. Numerical Analysis of Laterally Loaded Piles
Lyman C. Reese and Hudson Matlock, Associate Professors of Civil Engineering, University of Texas, Austin, Texas

Session C2—Ballrooms 3 and 4

Presiding: Sidney Shore, Chairman, Task Committee on Programming and Coding, Committee on Electronic Computation, Structural Division

9:00 A.M. A Computational Technique for Three-Dimensional Pin-Jointed Structures
Carl E. Pearson, Engineering Division, Arthur D. Little, Inc., Cambridge, Mass.

9:40 A.M. The Finite Element Method in Plane Stress Analysis
Ray W. Clough, Professor of Civil Engineering, University of California, Berkeley, California

10:20 A.M. Numerical Analysis Applied to Beam Vibrations
Edward N. Wilson, Ass't. Professor of Civil Engineering, New York University, New York, N. Y.

11:00 A.M. Inversion of Band Matrices
S. O. Asplund, Professor of Structural Mechanics, Chalmers University of Technology, Gothenburg, Sweden

FRIDAY AFTERNOON 10 SEPTEMBER 1960
Two simultaneous sessions

Session D1—Ballroom 2

Presiding: Elwyn H. King, Chairman, Task Committee on Educational Aspects, Committee on Electronic Computation, Structural Division.

2:00 P.M. Computer Analysis of Twin Box Culverts
James S. Hoffman, Programming Engineer, Iowa State Highway Commission, Ames, Iowa

2:40 P.M. Computer Analysis of a Pier with Unlimited Shape
Andrew R. Barkocy, Head of Computer Dept., Vogt, Ivers, Seaman & Associates, Cincinnati, Ohio

3:20 P.M. Computer Design of Prestressed Concrete Beams for Deflection Control
Felix Kulka, Associate, T. Y. Lin and Associates, Van Nuys, Calif.

Session D2—Ballrooms 3 and 4

Presiding: Chas. W. Zahler, Chairman, Task Committee on Statistical Applications, Committee on Electronic Computations, Structural Division

2:00 P.M. Computer Methods for Dynamic Structural Response
V. H. Neubert, Supervisor, Mechanics of Solids Group, Applied Mechanics Section, Electric Boat Division of General Dynamics Corp., Groton, Conn.

2:40 P.M. Matrix Analysis of Non-Linear Structures
Edward L. Wilson, Graduate Student, University of California, Berkeley, California

3:20 P.M. New Methods of Matrix Structural Analysis
Bertram Klein, Chief of Structures, Solar Aircraft Co., San Diego, California, and Matthew Chirico, Senior Research Engineer, Computing Laboratory, Convair-Astronautics, San Diego, California

In addition to the very fine technical program detailed above, there will be three addresses of general interest to the Conference participants. They are as follows:

Keynote Address

Impact of Electronic Computation on Civil Engineers
George S. Richardson, Senior Partner, Richardson, Gordon and Associates, Pittsburgh, Pennsylvania

Thursday Luncheon Address

A Vision of Our Automatic Future
Neal J. Dean, Partner, Booz, Allen & Hamilton, Chicago, Illinois

Friday Luncheon Address

The Use of Computers in Election Forecasting
Leon Nemerever, President, Computer Operations Inc., Garden City, New York

SYMPOSIUM ON PROBABILITY APPLICATIONS

A two-day Symposium on Engineering Applications of Probability and Random Function Theory sponsored by Purdue University will be held November 15-16, 1960, at Lafayette, Indiana. The symposium will stress applications of probability and random function theory to problems associated with factors of safety in structures, reliability of structures and systems, optimization of systems of all types which are subject to random disturbances, jet and rocket engine noise fields and traffic control.

Requests for further information should be addressed to either J. L. Bogdanoff or F. Kozin, co-chairmen of the Symposium, Division of Engineering Science, Purdue University, Lafayette, Indiana.

COMMITTEE ON ELECTRONIC COMPUTATION

In the 1960 ASCE Official Register, on p. 43, the listing for the Administrative Committee of the Committee on Electronic Computation is in error. The correct listing is as follows:

Nathan M. Newmark, Chairman, Sidney Shore, Vice Chairman, Glen V. Berg, Jackson L. Durkee, Elwyn H. King, John J. Kozak, James Michalos, Charles W. Zahler, Steven J. Fenves, Secretary, Nathan D. Whitman, Jr.*

STRUCTURAL DIVISION JOURNAL

The news item about the new Joint ACI-ASCE Committee on Prestressed Concrete was submitted by Prof. Phil M. Ferguson, Chairman, Committee on Masonry and Reinforced Concrete. Prof. Ferguson is a regular contributor of news items and deserves the thanks of the Division for keeping them informed about the work of his Committee through the medium of the NEWSLETTER.

The next issue of the Structural Division Newsletter will be published in November. Material for this issue must reach the editor before the end of September. Committee Chairmen are requested to send to the editor news items and minutes of any task committee meetings.

COMBINED INDEX TO ASCE PUBLICATIONS

For complete coverage of the Society's 1959 year in print, there is now a Combined Index covering the Division Journals, Transactions, and Civil Engineering. Also included are reprints of the Proceedings Abstracts that are published each month in Civil Engineering. The price of the Combined Index

(ASCE publication 1960-10) is $2.00 with the usual 50% discount to members. The coupon herewith will make ordering easy:

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